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The Commonwealth of Massachusetts
Department of Education

MATHEMATICS

CURRICULUM FRAMEWORK

Achieving Mathematical Power

Massachusetts
"We help children learn."
Department of Education

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The Commonwealth of Massachusetts
Department of Education
350 Main Street, Malden, Massachusetts 02148-5023 • 617•388•3300

DEAR COLLEAGUES:

I am pleased to present to you the Massachusetts Mathematics Curriculum Framework which was approved by the Board of Education in December, 1995. It represents the first statewide guideline for learning, teaching, and assessment in mathematics for the state's public schools.

This framework is based on sound research and effective practice, and reflects a vision of how classrooms can and should look to assist all students to achieve higher standards of excellence. It offers exciting opportunities to strengthen curriculum and instruction in grades PreK - 12 and in adult basic education programs across the Commonwealth.

I am proud of the work that has been accomplished. The thousands of comments and suggestions we received on our previous working drafts have strengthened this curriculum framework. I want to thank all of you for working with us to create a high quality document that provides challenging learning standards for our students. We will continue to work with you as you implement them in your district.

I encourage you to offer comments on the curriculum frameworks as you use them in schools to develop curriculum and programs. We want to keep them current and vibrant so that they will continue to bring education reform alive in our classrooms.

Thanks again for your ongoing support and cooperation.

Sincerely,

Robert V. Antonucci

Robert V. Antonucci
Commissioner of Education

Massachusetts
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Department of Education

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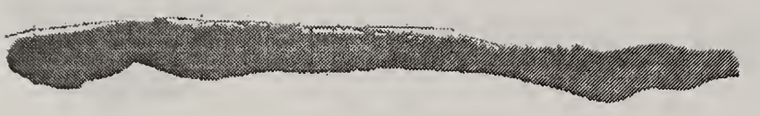

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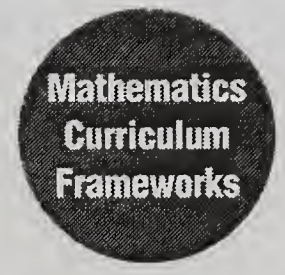

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Vignettes

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Reasoning about a pie chart, pages 92-93, Shelly Rasmussen and Regina Panassiouk, University of Massachusetts, Lowell; Patricia Pegram, Susan Daily, and Christine Burnham, Lawrence Public Schools

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Preface

Developing the Massachusetts Mathematics Curriculum Framework

Achieving Mathematical Power is based upon two reform initiatives in Massachusetts, the Education Reform Act of 1993 and Partnerships Advancing the Learning of Mathematics and Science (PALMS). PALMS is the Statewide Systemic Initiative, a collaborative effort jointly funded by the National Science Foundation and the Ccmmonwealth of Massachusetts, which began in 1992. Of the seven initial goals for this initiative, the first was to develop, disseminate, and implement curriculum frameworks in Mathematics and Science and Technology. In January 1993, a set of recommendations for PreK–8 frameworks was produced by consultants from TERC, a research and development corporation based in Cambridge. Their work was reviewed by the PALMS Curriculum Framework Advisory Committee, and formed the basis for the later drafts of the mathematics and the science and technology frameworks. With the passage of the Massachusetts Education Reform Act in June 1993 and additional funding from the U.S. Department of Education, development of the curriculum frameworks was extended to include grades 9–12 and Adult Basic Education.

The creation of the mathematics framework was a collaborative endeavor among members of the Framework Development Committee—teachers, school and district administrators, mathematicians, college faculty, parents, and representatives of business and community organizations across the state. A majority of the members are currently classroom teachers who have extensive experience teaching mathematics at elementary, middle, and high school levels. Committee meetings were convened to consider each draft framework from the standpoints of clarity, accessibility, consistency, pedagogy, mathematical correctness, and alignment with the Massachusetts Common Core of Learning and the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards.*

The core concept of *Achieving Mathematical Power* is that students develop mathematical power through problem solving, communication, reasoning and connections. The mathematics framework is more than a collection of concepts and skills. For each individual it involves methods of investigating and reasoning, means of communication, notions of context, and development of personal self-confidence. The framework provides quality and equity for all learners. The Guiding Principles and Habits of Mind of this framework outline ways in which this can become a reality.

The Mathematics Content section presents an outline upon which district and school curricula, instruction, and assessment can be developed. This section is organized into four strands, within which learning standards are articulated. Examples of student learning, vignettes, models, diagrams, and graphics contextualize and enhance the standards. It is a curricula for all that includes a broad range of content, a variety of contexts, and deliberate connections. The goals for all learners in the Massachusetts mathematics framework are that they value mathematics, become confident in their ability to know and to do mathematics, become mathematical problem solvers, and learn to reason and communicate mathematically.

The Massachusetts Mathematics Curriculum Framework

CORE CONCEPT

Achieving Mathematical Power through problem solving, communication, reasoning, and connections

Guiding Principles of Mathematics Education	
I	Students *explore mathematical ideas* in ways that maintain their enjoyment of and curiosity about mathematics, help them develop depth of understanding, and reflect real-world applications.
II	*All* students have access to high quality mathematics programs.
III	Mathematics learning is a *lifelong process* that begins and continues in the home and extends to school and community settings.
IV	Mathematics instruction both *connects* with other disciplines and moves toward *integration* of mathematical domains.
V	*Working together* in teams and groups enhances mathematical learning, helps students communicate effectively, and develops social and mathematical skills.
VI	*Technology* is an essential tool for effective mathematics education.
VII	Mathematics *assessment* is a multifaceted tool that monitors student performance, improves instruction, enhances learning, and encourages student self-reflection.

Habits of Mind: A Guide for Reflection

- Working beyond the center of competence
- Deepening mathematical understanding
- Gaining an attitude of persistence
- Integrating mathematics in a network of ideas
- Valuing all dimensions of mathematics

Content in Mathematics

Students will use Problem Solving, Communicating, Reasoning, and Connecting to explore, develop, investigate, and know . . .

Number Sense	Concepts of Whole Number Operations, Fractions and Decimals, Estimation, Whole Number Computation, Number Systems and Number Theory, Discrete Mathematics, and Mathematical Structure
Patterns, Relations, and Functions	Relationships, Algebra, Trigonometry, and Conceptual Underpinnings of Calculus
Geometry and Measurements	Spatial Sense and Geometry from an Algebraic Perspective
Statistics and Probability	Gathering and Graphing Data, Chance, Simulations, and Theoretical Probability

Adult Basic Education

This framework includes discussion of Adult Basic Education throughout each of its components. It features separate and distinct learning standards and examples of student learning for each of the strands.

The Beauty and Power of Mathematics

The harmony of the world is made manifest in form and number, and the heart and soul and all the poetry of natural philosophy are embodied in the concept of mathematical beauty.

Sir D'Arcy Wentworth Thompson

Mathematics is a universal language of numbers that is spoken in all cultures. It is an equalizer. Mathematics is beautiful patterns in nature and art, pleasing proportions in architecture. The form of poetry is mathematics and the shape of music is mathematics. It is systematic, random, and chaotic. Mathematics is about relationships. It brings joy in discovery and satisfaction in mastery. Mathematics is about exploration, intuition, and strategy.

Mathematics has an intrinsic value. It beckons each of us to appreciate and value mathematics for its own sake. When learners progress from intellectual struggle to achieving mathematical knowledge, the aesthetic value of mathematics emerges.

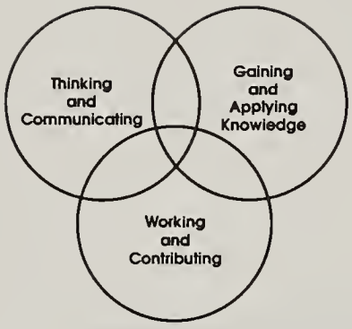
The vision of the Massachusetts Mathematics Curriculum Framework is that all students in the Commonwealth achieve mathematical power through problem solving, communication, reasoning, and connections. This vision of mathematical power exemplifies the principles in the Common Core of Learning. This framework sets forth as a goal that all students become effective problem solvers, master computation in the early elementary grades, and master basic algebraic concepts by the end of eighth grade.

Problem solving is the central focus of mathematics education. Whenever we apply our mathematics knowledge, skills, or experiences toward resolving a dilemma or situation that is new or perplexing, we are problem solving. Whether we are adding and subtracting to make change, using proportional reasoning to build a snowman, calculating distance to choose the best route home, or exploring patterns in music to choreograph a dance, we are engaging in mathematical problem solving. To become good problem solvers, students need many opportunities to create and solve problems in both mathematical and real-world contexts. They also need to develop and build upon a strong number sense. As stated in the Massachusetts Common Core of Learning, students "develop, test, and evaluate possible solutions." This involves posing questions, defining problems, considering different strategies, and finding appropriate solutions.

"Mathematical power includes the ability to explore, conjecture, and reason logically; to solve non-routine problems; to communicate about and through mathematics; and to connect ideas within mathematics and between mathematics and other intellectual activity. Mathematical power also involves the development of personal self-confidence and a disposition to seek, evaluate, and use quantitative and spatial information in solving problems and in making decisions. Students' flexibility, perseverance, interest, curiosity, and inventiveness also affect the realization of mathematical power."

National Council of Teachers of Mathematics

Communication of mathematics reflects mathematical understanding and embraces mathematical power. The Common Core of Learning suggests that all students should ". . . justify and communicate solutions to problems." Students learn mathematics as they talk and write about what they are doing. They become actively engaged in doing mathematics when they are asked to think about their ideas, or talk with and listen to other students, sharing ideas, strategies, and solutions. Writing about mathematics encourages students to reflect on their work and clarify ideas for themselves. Reading what students write is an excellent way for teachers to identify students' understanding and misconceptions.



Mathematical reasoning is necessary if we are to know and do mathematics. The ability to reason enables students to solve problems in their lives, inside and outside of school. Whenever we use reasoning skills to validate our thinking, we enhance our confidence with mathematics and mathematical thinking. Mathematics as a field of study is characterized as much by particular types of reasoning as by particular types of content. The Common Core of Learning states that "all students . . . should make reasoned inferences and construct logical arguments." These reasoning skills suggest an ability to recognize and use deduction and induction, as well as develop and apply these skills in numerical and spatial contexts.

Connections ground mathematics in daily life and provide the synapses through which we relate mathematical topics with one another. Students should understand how mathematics relates to other subject areas—the arts, social studies, health, science and technology, world languages, and English language arts. For example, the Common Core of Learning suggests that "all students . . . should understand concepts such as location and place . . ." This exemplifies one of the interdisciplinary relationships that exists between geography and mathematics. Students appreciate that mathematics is one way of learning about the world, and that it is connected, not isolated, from other ways of learning. Individuals' strengths and interests enhance those subject areas with which they feel less comfortable. For example, a proficient mathematics student may begin to think of himself as a talented arts student when he explores drawing through the study of perspective in mathematics class. Or, a history buff may gain new insight to her potential when she analyzes data from a questionnaire on family histories.

"Profound thoughts arise only in debate, with a possibility of counterargument, only when there is a possibility of expressing not only correct ideas but also dubious ideas."

Andrei Dmitrievich Sakharov

The atmosphere Massachusetts teachers create in their classrooms encourages students to ask questions or propose partial ideas, thereby building students' confidence in their mathematical abilities. Students in elementary grades begin working on complex mathematical ideas in informal ways as they are learning computational skills. This provides contexts for computation as well as laying the ground work for future mathematics learning. Students try uncharted ways of thinking and draw upon a repertoire of problem-solving strategies. They often work in teams, testing and verifying their ideas. No longer do they look only to the teacher for the final validation. Students learn from each others' explanations. They clarify their own understanding as their peers question their findings, enhancing the learning process.

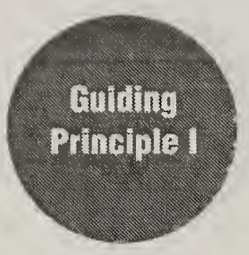
Massachusetts needs all of its citizens to achieve mathematical power. With effective mathematics education today, tomorrow's leaders can call upon more empowered citizens, who will have the resources to make informed decisions, the curiosity to explore new ideas, and the perseverance to face obstacles. They will be mathematically empowered with an adeptness to solve problems, a common language with which to communicate with other citizens and cultures, the self-confidence required for effective reasoning, and an understanding of the connections among mathematical ideas and everyday life.

Third-grade students work together, gathering and analyzing measurement data, as an educator observes.

Guiding Principles

These are the underlying beliefs and tenets central to the vision of mathematical power and content standards for mathematics education in Massachusetts.

Guiding Principles of Mathematics Education
I Students ***explore mathematical ideas*** in ways that maintain their enjoyment of and curiosity about mathematics, help them develop depth of understanding, and reflect real-world applications.
II ***All*** students have access to high quality mathematics programs.
III Mathematics learning is a ***lifelong process*** that begins and continues in the home and extends to school and community settings.
IV Mathematics instruction both ***connects*** with other disciplines and moves toward ***integration*** of mathematical domains.
V ***Working together*** in teams and groups enhances mathematical learning, helps students communicate effectively, and develops social and mathematical skills.
VI ***Technology*** is an essential tool for effective mathematics education.
VII Mathematics ***assessment*** is a multifaceted tool that monitors student performance, improves instruction, enhances learning, and encourages student self-reflection.



Guiding Principle I

Students explore mathematical ideas in ways that maintain their enjoyment of and curiosity about mathematics, help them develop depth of understanding, and reflect real-world applications.

Though math learning follows a certain progression, it is not a purely linear process, but is recursive, with children needing to rediscover and refine "old" concepts and skills as they build "new" ones. Also, as with all learning, the development of math skills is unique to each child.

Early Childhood Today

Developing mathematical power is a complex process. The mathematics that students learn depends not only on what is taught, but also on how it is taught. Curriculum cannot be separated from the instructional practices used to teach it. Instructional strategies should encourage students to engage intellectually with important mathematical ideas, to embrace the aesthetic value of mathematics, and to use mathematical principles to solve problems in their daily lives.

Teachers must have high expectations for all students and be familiar with each student's knowledge base in order to plan developmentally appropriate work. The intellectual, social, and emotional development of students should guide our choices of mathematics experiences for our students.

- What experiences has this student encountered at home, on the playground, in the cafeteria or classroom, or in earlier years that affect how the student learns?
- What is the learner's background in mathematics education? What mathematics knowledge has the learner acquired? What challenges the student? Has the learner overcome obstacles with aspects of mathematics?

The following vignette depicts how a teacher explores ratio with his students at different developmental stages.

> Mr. Manning plans to explore ratio with his middle grades mathematics class. He knows some of the students are not developmentally ready to grapple with proportionate reasoning in terms of ratio and percent, while others have begun to explore these concepts of number sense and relationships in the context of data analysis. Knowing that all students have developed an understanding of fractional expressions of number, Mr. Manning chooses to approach ratio and percent through a probability investigation based on the hand game of Paper, Scissors, Rock. This provides accessibility for all students, while respecting their individual developmental levels.

Experimenting with ideas, inventing constructs, and exploring our curiosities are at the foundation of all learning, including mathematics. Asking the right questions of students promotes creative thinking, prompting them to look deeper into their imaginations. Students can be encouraged to reflect on their learning and articulate their reasoning through questions such as:

- How did you work through this problem?
- Why did you choose this particular strategy to solve the problem?
- Are there other ways? Can you think of them?
- How can you be sure you have the correct solution?
- Could there be more than one correct solution?
- How can you convince me that your solution makes sense?

A rich matrix of ideas should be explored thoroughly throughout each academic year. Students should have regular opportunities to revisit important mathematical ideas throughout each school year and from one year to the next.

If students are to develop mathematical understanding, then they should engage in tasks of inquiry, reasoning, and problem solving that reflect real-world mathematical practice. In addition, hands-on exploration can deepen understanding of abstract concepts by encouraging the practice of process skills and communication, and allowing for reflective thinking. For example, when students explore prime numbers, they may engage in a manual approach such as the Sieve of Eratosthenes, then extend their understanding by modeling prime and composite numbers when they build rectangular arrays with cubes, as well as communicate and reflect upon their ideas about prime numbers when they create human pyramids. When students later use technology to explore number theory, they have already acquired a knowledge base of the concept.

Students learn best when they can connect their classroom learning to real-life experiences, and when they can experience the same concept or idea in multiple contexts. For example, students often make place value errors when working with a multiplication algorithm. To address this, another approach to learning the algorithm would be exploring multiplication by means of students' using place value blocks to represent area, then using addition to add the units, rods, and flats to measure area, thereby inventing for themselves the multiplication algorithm and discovering the meaning and importance of place value in mathematics.

The vision for mathematics in Massachusetts will become reality only if students deepen their understanding of mathematics by means of activities, investigations, and projects that promote inquiry, discovery, and mastery. For example, students may make pinwheels to explore rotational symmetry. A group might canvas the cafeteria, collecting data from the student body to test a hypothesis. Real-life meaning and mathematical structure will permeate the mathematics curriculum.

Investigations that a teacher introduces to a class should target important mathematical ideas, illuminate the connections among mathematical ideas, and identify relationships between the ideas introduced in the investigation and the concepts with which students are already familiar. It makes sense to embed problems or investigations in and draw resources from the various cultures and backgrounds of students. The questions that follow are suggested as one way to help teachers plan investigations.

Questions to consider when planning an investigation:

- Have I *identified* and *defined* the mathematical content of the investigation, activity, or project?
- How does the investigation *address the learning styles* and diverse *backgrounds* of all students within the classroom?
- Do I have a plan to *initiate thoughtful discussion* of or *reflection* on the concepts explored and of the relationships uncovered in the process to help students clarify their understanding and integrate it fully into their existing network of ideas?
- Have I *carefully compared* the network of ideas included in the curriculum with the students' knowledge?
- Have I allowed time to *note discrepancies, misunderstandings, and gaps* in students' knowledge as well as evidence of learning?
- How is the investigation designed to *test* students' false assumption, *confirm* accurate findings, and *extend* the students' knowledge?

Guiding Principle II

All students have access to high quality mathematics programs.

"My mom and I can't do math."

Alma, a second grade student from Dorchester

"Let's get together and talk about math."

Alma's teacher to Alma's mother

All students must have access to high quality mathematics programs that support successful learning of mathematics and help them to develop a mathematical sense and intuitive understanding. It is our responsibility to ensure that students are fairly represented in each mathematics program and have equitable access to resources. Everyone learns best in an environment that acknowledges, respects, and accommodates each learner's background, learning style, and gender. For example, a teacher's listening attentively to all students' ideas helps to foster in students a sense of control of their future. Or, by making special efforts to achieve classroom integration when students self-segregate, a teacher enhances students' respect for others' backgrounds and learning styles.

This vignette shows how one Massachusetts teacher seized the moment to explore ratio in the context of gender equity. She relates:

> When students were surprised that gender prejudices existed, I was relieved. I suggested that we conduct some kind of investigation to see whether their perceptions of my lack of bias were accurate.
>
> We decided that a sample of just that one classroom would not suffice. Rather, the entire grade comprising ninety-two students became involved. For two weeks, students monitored the number of girls versus the number of boys on whom I called. Because there were so many more boys than girls in one class, they were quick to decide that ratios were needed (daily, because of absentees). I capitalized on this for a quick review of ratio fundamentals!
>
> When it seemed that I, indeed, may have been favoring the boys, I wanted my students to know that it was unacceptable to me. Gradually, I noticed that there was a more balanced distribution of raised hands between the boys and girls.

"The underrepresentation of certain groups in mathematics, science, and technology education is well documented. The Statewide Systemic Initiative seeks to effect change in student participation by transforming school curriculum, classroom instruction, teacher education, and by increasing parental support so that all students have opportunities suited to their needs to learn mathematics, science, and technology."

National Science Foundation,
Equity Framework in Mathematics, Science, and Technology Education

All students should see themselves as mathematicians, capable of using their evolving mathematical power to solve new problems. Some students may ultimately progress further in their mathematical learning than others, and learning may take different amounts of time for different learners. However, if each student is offered an accessible approach to learning mathematics, consistent with his learning style and experience, then all students can learn mathematics. This means establishing high standards of expectations and helping students when they are struggling with mathematics. It is not enough to enroll students in higher level classes; everything possible must be done to engage their interests. Each member of every class should participate meaningfully.

The diversity in communities and classrooms should be treated as an advantage that can help all learners in Massachusetts schools. The presence of diverse learners in Massachusetts classrooms presents an opportunity for all students and teachers to learn about the rest of the world and appreciate the talents and culture of each individual. Since different cultures sometimes use alternative mathematical strategies or perceive the relationships of objects and events in the world in ways other than the mainstream culture, their strategies and understandings can enrich the understanding of all students. For example, Cambodian children learn a different algorithm for division. If given the opportunity to explain their method to the rest of the class, then everyone broadens their cultural experiences, deepens their understanding of the concept of division, and recognizes the varied approaches to mathematics.

If the women, minorities, and individuals with disabilities who have made important contributions to the field of mathematics and embarked on careers that utilize mathematics are presented as role models, then all students see for themselves its practical applications. Knowing that increased opportunities await students who are mathematically empowered, families, educators, and communities should encourage students to continue their mathematics learning through grade twelve. Guidance counselors, students, and families should be fully aware of the impact that mathematics has upon future access to higher education and employment opportunities.

Guiding Principle III

Mathematics learning is a *lifelong process* that begins and continues in the home and extends to school and community settings.

The need to make sense of the world begins before kindergarten and continues beyond formal schooling. PreKindergarten students begin to form ideas about mathematics as part of the natural process of exploring their world. Building with blocks gives them an opportunity to begin developing an understanding of shape, size, position, and symmetry. Gathering items such as rocks, shells, toy cars, or erasers for their collections leads to discovery and exploration of patterns and classification. Such informal explorations are important developmental precursors to an understanding of mathematics. They are the beginning of lifelong skills that enable us to learn more abstractly.

Preschool activities give children opportunities to solve problems and discover information for themselves in an environment where they can explore freely and safely. Young children and school-age students need repeated experiences exploring materials, time to talk about their experiences, and freedom to experiment and learn from each other. In addition to the mathematics that students learn in PreKindergarten through grade twelve classrooms, there are other arenas in which the learning of mathematics can be strengthened.

For example, families explore mathematics together, as described in this vignette.

> Teddy is working with his grandmother, sorting laundry. Teddy picks up several socks and puts them in a row. His grandmother says, "Teddy, you're sorting socks!" Teddy replies, "These are the Three Bears' socks — Mamma Bear's sock, this big one is Papa Bear's and this one is Baby Bear's!" Grandmother says, "I need to sort these socks to put them away. Help me sort them." His grandmother holds up the 'Mamma Bear's sock' and Teddy finds a matching one and they fold together. Teddy and his grandmother continue until all the socks are sorted into pairs. They count the pairs of socks they have sorted. Later on in the morning, Teddy's grandmother reads aloud Teddy's favorite story, *Goldilocks and the Three Bears*.
>
> Later on in the day, Teddy continues sorting laundry with his grandmother. Teddy picks up several socks of many different sizes. He starts placing them randomly on the floor—large red sock and small blue sock. His grandmother suggests another way to sort the socks. "I wonder if we can find all the socks that are the color red," she says as she holds up a red sock. Teddy helps find five other red socks. He then counts the red socks on the floor: "One, two, three . . ." Later in the afternoon, Teddy's grandmother reads aloud *One Fish, Two Fish, Red Fish, Blue Fish* by Dr. Seuss.

Community programs, museums, and businesses can provide valuable learning experiences that enhance and extend classroom activities. Careful planning in these contexts will provide opportunities for developing students' curiosity, creating skepticism, and promoting self-esteem, as shown in the following vignette.

"Learning about math, science, social studies, health, and other content areas are all integrated through meaningful activities such as those when children build with blocks; measure sand, water, or ingredients for cooking; observe changes in the environment; work with wood and tools; sort objects for a purpose; explore animals, plants, water, wheels and gears; sing and listen to music from various cultures; and draw, paint, and work with clay."

National Association for the Education of Young Children

Mrs. Jones, who teaches both science and mathematics to her sixth grade class, is beginning a unit on models. She wants to work with her students on proportion and scale and decides to take a trip to the Museum of Science to introduce these ideas to her class. Although Mrs. Jones is familiar with the museum, she has never used it to study models, so she visits the museum before the field trip. (Massachusetts teachers have free admission and parking privileges.) She finds the museum full of models—living organisms and mathematical, physical, life-sized, and scaled models—everything from mollusks to machines.

Before the trip, Mrs. Jones introduces the concept of models to the class and asks her students to make a list of familiar models. She asks the students to think of models they have seen in the museum on previous trips, and continues the discussion by posing questions such as: What are the differences between models and real objects? Can models be misleading? When are scale models useful? Is there a change in structure when a model is scaled down or up?

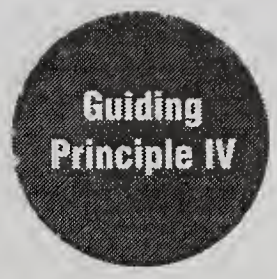
At the museum, students are asked to find examples of a real object, a larger than life model, a life-sized model, a smaller than life model, a physical model, and a mathematical model. Students then create lists of what the models do and do not tell them about the object. They also respond to a series of questions. Does the model suggest something that might not be true (e.g., a globe of the earth in which the height of the mountains is in a different scale from the diameter of the planet)? Why is the model portrayed in a particular scale? Can the scale be misleading?

Back in class, students compare their findings and discuss their observations and ideas. At the end of the unit, each student uses clay to create three scale models of an object: life-size, one-half life size, and twice life-size. Mrs. Jones uses this project to assess the students' knowledge and understanding of models and scales.

"The learners' inquiries into the formation of graphs lead to more work . . . they were instructed to bring in graphs of interest to them . . . . There were bar graphs on everything from consumption of candy bars per person per year, to circle and bar graphs on the concentration of wealth in the United States . . . and most important, [on] how the less educated were suffering more. This last one had a great impact upon my learners, for they noted the importance of a high-school diploma and its connection to a better income."

Adult Basic Education Math Standards Project

Adult mathematics learners often seek further education to meet a specific goal, perhaps to advance their career, or to help their children, or for self improvement. This framework supports them by offering hands-on problems based on real-life situations, using a range of manipulatives and technologies. Adult learners should be active participants in defining personal learning objectives and deciding measures of success. Instruction should include opportunities to question, discuss, and write about ideas.



Guiding Principle IV

Mathematics instruction both *connects* with other disciplines and moves toward *integration* of mathematical domains.

Upon this gifted age, in its dark hour,
Falls from the sky a meteor shower
Of facts . . . they lie unquestioned, uncombined.
Wisdom enough to leech us of our ill
Is daily spun; but there exists no loom
To weave it into fabric . . .

Edna St. Vincent Millay

To help us understand the world, we draw upon a knowledge base that spans disciplines and experiences, forming networks of thoughts and ideas. Students may relate their knowledge of functions to mechanics, patterns to music, or statistics to the economy in Massachusetts. In all cases, they are connecting their understanding of mathematics with other disciplines and their world.

A move toward an integrated approach within the mathematics curriculum is recommended. Beginning with PreKindergarten, an integrated approach to mathematics might include activities that combine sorting, measurement, estimation, and geometry. In middle schools and high schools, it will mean helping students make connections between algebra and geometry, but also among ideas from discrete mathematics, statistics, and probability. In Adult Basic Education, it will mean establishing connections between mathematics and daily life at home, at work, and in the community.

An integrated mathematics curriculum gives students a more accurate picture of the nature of mathematics, contextualizing the essential connections among various fields of mathematics. Integrating the domains encourages students to approach problem solving in more than one way, making them more powerful problem solvers. Students will be able to solve problems numerically, algebraically, and graphically. The use of technology and computer software facilitates connections and integration. For example, graphing calculators make it possible to switch from equations to graphs to data analysis. If a problem can be approached either visually or numerically, then it may be more accessible to a visual learner struggling with abstraction. As solutions are shared, these same visual learners will have the opportunity to explore the problem numerically.

Furthermore, an integrated mathematics curriculum reduces review time, since students will continually encounter ideas from all parts of mathematics. More time will be available to teach recommended mathematics topics, such as statistics and probability, which have not been part of the traditional mathematics curriculum for all students.

"Socrates advocated reflection as opposed to observation, an activity dependent upon a principle that is important to any theory of reflective method: what we are trying to do is not to discover something of which until now we have been ignorant, but to know better something which in some sense we knew already; to know it better in the sense of coming to know it in a better and different way."

Paul Bitting and Renee Clift

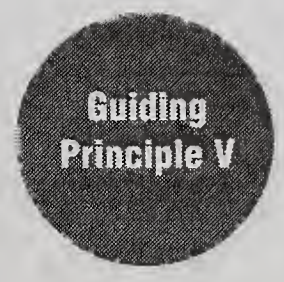
Images of Reflection in Teacher Education

Part of the task for schools and districts will be to develop an approach to a more integrated curriculum that is appropriate for the local circumstances. The following vignette shows how it worked in one high school in the Commonwealth.

> During the 1991-1992 school year, a team selected new integrated textbooks. Members considered two critical questions during their evaluation. The first consideration was, did the textbook include a variety of examples and applications at many different levels so that students could proceed from simple to more complex problem-solving situations? And second, were algebra and geometry truly integrated rather than presented alternately, like a layer cake, within the text?
>
> Since then, teachers and students use a variety of approaches to explore mathematical ideas. For example, when working with the Pythagorean theorem, students do a hands-on proof, actually building squares on the legs of right triangles. They explore the theorem algebraically and by using a geometric proof. Teachers usually present for about five minutes at the beginning of class. Students work on problems both in cooperative groups and alone. Often they work with a partner, and then share results in a foursome. Students are asked to write and explain their thinking. At times, a group project is required.
>
> Assessment in this high school is often unorthodox. For example, one year students made a design that used several different figures, computed the area of each figure, and recorded how they tackled the problem. Although it took one week for teachers to review the projects, they gained many insights about students' understanding. In the cases where teachers could not understand a student's work, the student was asked to explain it.

Part of what made the integration efforts successful at this school is a history of open communication. Teachers are willing to observe and be observed by one another. They discuss individual students among themselves and with others, such as resource teachers. The results of the teachers' efforts are an improved mathematics program for the students and professional growth for the teachers.

Middle grade students explore the science of bubbles while studying surface area in mathematics class.



Guiding Principle V

Working together in teams and groups enhances mathematical learning, helps students communicate effectively, and develops social and mathematical skills.

Independent learning helps to develop self confidence. However, students *deepen* their understanding of mathematics as they interact with the ideas, theories, and opinions of their peers and teachers. Being able to communicate mathematical ideas in a variety of ways helps students to "develop, test, and evaluate possible solutions" as suggested in the Common Core of Learning. Team work encourages members to interact with others, enhances self-assessment, encourages the exploration of multiple strategies, and helps prepare students to be members of the workforce. The variation offered by group work leads to enriched solutions and offers an ideal venue for informal assessment. Here is a vignette that describes one teacher's use of team work in elementary school.

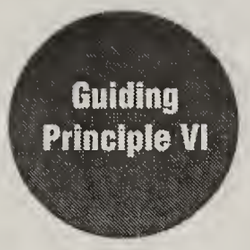
> Ms. Gutierrez' third grade class is enthusiastic this morning as students actively learn division facts. One group is working diligently with base ten blocks, discovering patterns in division. In the opposite corner, another group is comfortable on the reading rug, brainstorming ideas to devise a rule for using division by ten and one hundred to help them solve problems mentally. After lunch, students from both groups will pair up and present their findings to each other.

A teacher may group students in mathematics for a number of reasons — students may be working together on different projects, the teacher may work with a group of students who need special help, or highly motivated students may be working on enrichment activities. Some groupings may be temporary and some may be of longer duration.

One of the most complex and difficult tasks for teachers, schools, and districts will be how they group students in order to promote the achievement of mathematical power for all students. Here are some considerations:

- High expectations and standards should be established for all students, including those with gaps in their knowledge bases.
- Students should be encouraged to achieve their highest potential in mathematics.
- Students learn mathematics at different rates, and different students' interests in mathematics vary.

Support should be available for all students based on individual needs. Appropriate opportunities for enrichment and advancement need to be provided for students at all achievement levels. The mathematics should be challenging and expectations for all students should be high.



Guiding Principle VI

Technology is an essential tool for effective mathematics education.

All students should use computers and other technologies to obtain, organize and communicate information and to solve problems.

Massachusetts Common Core of Learning

"When technology is implemented in our schools, change occurs. Electronic formats enable access to information from almost anywhere in the world, crossing age groups and cultures with ease .... Students move into the world possessing the skills to gather, manage, and assimilate the vast resources of information at their fingertips."

***The Switched-on Classroom,* Massachusetts Software Council**

Students in all courses need access to tools for learning mathematics. These tools include measuring instruments, manipulatives, graphing calculators, and computers. The calculation powers of computers and calculators free students from the rote tasks of computation but do not replace the thinking that underlies mathematical operations. Computer software for modeling and visualization of mathematical ideas such as statistics and probability or fractals and chaos can open a whole new world to students and help them connect these mathematical ideas to their language and symbol systems.

> Students at one Massachusetts high school explore open-ended lab projects, using a geometry construction software program that enables them to collect data, make observations, and develop conjectures. They must support their findings by writing lab reports that summarize the exploratory process and conclusions they have drawn. This makes the transition to deductive proof a much more natural extension of the learning process. Students are encouraged to construct and justify their own understanding of the geometry explored when they participate in a follow-up discussion, facilitated by the teacher. [1]

Technology tools, when integrated in a mathematics program, raise the level of mathematics to which students can be exposed, improve their self-confidence, and facilitate increased student–teacher interactions. The availability of calculators, computers, and other technology has changed forever the way that people are able to think about and do mathematics. New technologies have changed our culture into an "information society." These changes have "transformed both the aspects of mathematics that need to be transmitted to students and the concepts and procedures they must master if they are to be self-fulfilled, productive citizens." [2] Some mathematics becomes more important because technology requires it, some becomes less important because technology replaces it, and some becomes possible because technology allows it. This integration of technology in our global society today impacts the lives of our students tomorrow.

For students with special needs, technology can be especially helpful in assisting students in regular and special classrooms, the home, and the community. New software, hardware, and assistive devices can all be used to help students succeed. Technology can enhance a student's access to the curriculum, increase opportunities to interact with peers, and increase avenues of communication.

Guiding Principle VII

Mathematics assessment in the classroom is a multifaceted tool that monitors student performance, improves instruction, enhances learning, and encourages student self-reflection.

"The assessment should examine the extent to which students have integrated and made sense of information, whether they can apply it to situations that require reasoning and creative thinking, and whether they can use mathematics to communicate their ideas. "

National Council of Teachers of Mathematics

The goal of classroom assessment is to provide information about students' evolving mathematical understanding, skills, and knowledge, so that teachers can give feedback to students on their progress and teachers can make decisions about where to go next with their instruction. Mathematics assessment has been primarily short answers to short questions. While this is one method of evaluation, a broader interpretation of mathematics suggests that assessment must take on many new dimensions.

Performance-based assessments are especially congruent with the goals of mathematics instruction. The three types of performance-based assessments discussed here are open-ended written assessments, portfolios, and observation.

Open-ended written assessments present questions to students that invite multiple approaches to problems, allow for creative expression of mathematical ideas, and encourage comparative analysis and reflection. By soliciting written responses, students are encouraged to communicate their strategies, develop their hypotheses, and explain their solutions by using prose, graphs, or drawings. An example of an open-ended question can be as simple as asking students to explain their reasoning or to justify their answers, or as multifaceted as asking them to design and conduct their own probability experiments.

Portfolio assessments imply that teachers have worked with students to establish individual criteria for selecting work for placement in a portfolio and judging its merit. Charts, models, constructions, and students' reflections on their work can all be included within mathematics portfolios. The contents of a mathematics portfolio should be indicative of each student's abilities and understanding, representative of his efforts, and indicative of progress over a period of time.

Observation as a means of assessment reveals a student's appreciation of mathematics, or the strategies he commonly employs to solve problems, or perhaps his preferred learning style. Formal observations are pre-planned, target specific mathematical skills, and refer to established criteria. These criteria for performance are usually expressed within a range and made known to the student.

Habits of Mind

Developing mathematical power includes the gradual acquisition of ways of thinking and behaving called mathematical habits of mind. These habits might be thought of as desirable attitudes toward learning and using mathematics that should become an integral part of each student's approach to mathematics. They derive from the curriculum and are reinforced when teachers model these habits for students.

The Mathematics Habits of Mind are listed here in the form of questions to be considered and modeled by teachers and shared with students. They help frame the vision of achieving mathematical power through problem solving, communication, and connections, which is the foundation of the Massachusetts Mathematics Framework.

Habits of Mind: A Guide for Reflection

- *How do I use my mathematical skills to interpret information and solve problems?* Mathematics can provide a venue for us to work beyond our center of competence, encouraging us to push the limits of our mathematical knowledge and abilities.
- *Do I communicate to others how I solved a problem or justified my solution?* Seeking feedback helps us to deepen our mathematical understanding by reflecting upon, extending and refining our thinking.
- *In what ways do I reflect confidence in my ability to do mathematics?* Confidence in mathematical ability brings with it an attitude of persistence when solutions are not apparent.
- *In what ways do I explore the relationship of mathematics to other areas that interest me? To other subject areas?* When we explore mathematics in the course of our daily lives, we encourage the use of available resources as well as integrate mathematics within our existing network of ideas.
- *How do I show that I value and appreciate the beauty and fascination of mathematics?* Valuing all dimensions of mathematics encourages us to view mathematics in unconventional ways, generating new ways of thinking.

Mathematics Content

The core concept of this Massachusetts Mathematics Framework is that students develop mathematical power through problem solving, communication, reasoning, and connections. Consider how incorporating *problem solving* in a mathematics curriculum provides the context in which students work together and communicate with one another. Or, imagine ways students *communicate* mathematically, providing opportunities to explain their reasoning, and to listen to and understand the *reasoning* of others. Think of a student's explaining his reasoning about fractals to a friend, then discovering that his solution is based on a diagram while his friend's solution is grounded in number patterns—this signifies *connections* between geometry and algebra.

An effective mathematics curriculum is one in which the skills and knowledge of *problem solving, communication, reasoning,* and *connections* are subsumed. The pages that immediately follow present models of what teachers, administrators, and districts are working toward in mathematics education, which is contrasted with what they are moving away from.

The standards set forth by the National Council of Teachers of Mathematics (NCTM) and The Massachusetts Adult Basic Education Math Standards that derive from the core concept of this framework are also presented. Each of these standards should be an integral part of each district's mathematics curriculum.

The four content strands in this framework are Number Sense; Patterns, Relations, and Functions; Geometry and Measurement; and Statistics and Probability. Each content strand represents a grouping of the NCTM standards. A chart showing the groupings is found in the appendix. It is important to understand that the four components of the core concept (problem solving, communication, reasoning, and connections) are *embedded* and *cross-cutting* within and among content strands. That is why the components of the core concept are found here, in the mathematics content chapter. They cannot be separated from the conceptual understanding and procedural knowledge that are also a part of mathematics education.

Following the discussions of problem solving, communication, reasoning, and connections are the Learning Standards, which identify what students should know and be able to do across grade spans. Examples of Student Learning and How It Looks in the Classroom contextualize some of the learning standards for the reader. They are not, however, intended as prescriptive or isolated activities.

"Mathematical power is conceived as consisting of mathematical abilities (conceptual understanding, procedural knowledge, and problem solving) within a broader context of reasoning and with connections across the broad scope of mathematical content and thinking. Communication is viewed as both a unifying thread and a way for students to provide meaningful responses to tasks."

Mathematics Framework for the 1996 National Assessment of Educational Progress
U.S. Department of Education

Problem Solving

PreK–12 and Adult Basic Education

"Problem solving is not a distinct topic, but a process that should permeate the entire program and provide the context in which concepts and skills can be learned."

National Council of Teachers of Mathematics

Here is a model of the paradigm shift that summarizes mathematics education reform.

Moving away from...	Working toward...
...solving routine word problems ...problems that are computational exercises embedded in a simple context. ...problem solving as doing problems following introduction and practice of a specific procedure. ...assuming that complex problems are appropriate for some students, and too difficult for other students.	...problem solving as the focus of mathematics programs. ...recognizing that *all* students are capable of solving problems, and are thereby given many opportunities to solve problems. ...students' developing and using a variety of approaches and strategies when solving problems. ...students' using multiple strategies to solve the same problem. ...students' realizing that there are multiple pathways to solutions as well as multiple solutions to some problems.

For example...

elementary students explore area and perimeter by creating solar panel roof designs by guessing, estimating, and checking with the aid of a geoboard, making a drawing, modeling with geometric tiles, or using a computer program.

Problem Solving

Below are listed the standards that exemplify **mathematics** as **problem solving.**

PreK-12 Standards	Adult Basic Education Standards
All students will... • Use multiple approaches to investigate and understand mathematical content. • Formulate problems from everyday and mathematical situations. • Develop and apply strategies to solve a wide variety of problems, including multistep and non-routine problems. • Verify and interpret results with respect to the original problem. • Generalize solutions and strategies to new problem situations. • Acquire confidence in using mathematics meaningfully. • Recognize and formulate problems from situations within and outside mathematics. • Apply the process of mathematical modeling to real-world problem situations.	*All students will...* • Explore and use multiple strategies for solving problems. • Determine, collect, and analyze appropriate data with respect to the original problem or in new problem-solving situations. • Access and use appropriate problem-solving tools, including calculators, computers, and measurement devices. • Generalize problem-solving strategies to a wide range of adult oriented, real-world situations.

Communication

PreK–12 and Adult Basic Education

"All students should develop and present conclusions through speaking, writing, artistic, and other means of expression."

Massachusetts Common Core of Learning

Here is a model of the paradigm shift that summarizes mathematics education reform.

Moving away from...	Working toward...
...the primary means of communicating mathematics as one-way—from the teacher (or textbook) to the students. ...students' solely communicating their knowledge of mathematics by using taught procedures in response to questions on tests. ...an emphasis on students' memorizing mathematical definitions as the method of internalization.	...students' learning by means of reflecting on ideas and communicating their thoughts with others. ...students' interacting with other students to solve problems, share strategies, and seek solutions to problems. ...students' explaining and justifying their mathematical thinking in writing. ...students' exploring mathematical terms by comparing and contrasting examples with other students before presenting a formal definition.

For example...

high school students make designs in coordinate planes and challenge partners to describe them mathematically, using systems of linear inequalities; and descriptively, using English or another language.

Communication

Below are listed the standards that exemplify **mathematics as communication.**

PreK-12 Standards	Adult Basic Education Standards
All students will... • Relate physical materials, pictures, and diagrams to mathematical ideas. • Reflect on and clarify thinking about mathematical ideas and situations. • Relate everyday language to mathematical language and symbols. • Use the skills of reading, listening, and viewing to interpret and evaluate mathematical ideas. • Model situations by using oral, written, concrete, pictorial, graphical, and algebraic methods. • Develop mathematical ideas, formulate mathematical definitions, and express generalizations discovered through investigations. • Ask clarifying and extending questions related to mathematics students have read or heard about. • Appreciate the economy, power, and elegance of mathematical notation and its role in the development of mathematical ideas.	*All students will...* • Develop the appropriate reading, writing, listening, and speaking skills necessary for communicating mathematically in a variety of settings. • Discuss mathematics with others, reflecting and clarifying individual thinking about mathematical outcomes. • Make convincing arguments and informed decisions. • Define everyday, work-related, or test-related mathematical situations by using concrete, pictorial, graphic, or algebraic methods. • Appreciate the value of mathematical language and notation in relation to mathematical ideas.

Reasoning

PreK–12 and Adult Basic Education

"If we would guide by the light of reason, we must let our minds be bold."

Louis Dembitz Brandeis

Here is a model of the paradigm shift that summarizes mathematics education reform.

Moving away from...	Working toward...
...some students investigating and applying mathematical reasoning, while others learn mathematics solely by memorization of rules and procedures.	...*all* students reasoning about mathematics, responding to the reasoning of others, and communicating their reasoning to others.
...developing mathematical understanding through teacher explanations, and only elementary students use manipulatives as tools for learning.	...students making conjectures, thinking about and selecting sensible ways to solve problems, and justifying their solutions.
...the notion that basics must be mastered before proceeding to higher level mathematics.	...students recognizing different types of reasoning as they use them.
...the belief that most students are not mature enough to do complex or abstract reasoning.	...students understanding the degrees of certainty associated with deductive, inductive, analogic, and statistical reasoning.

For example...
...students find examples of misguiding statistics in the media.

Reasoning

Below are listed the standards that exemplify mathematics as reasoning.

PreK-12 Standards	Adult Basic Education Standards
All students will... • Draw logical conclusions about mathematics. • Use models, known facts, properties, and relationships to explain mathematical thinking. • Justify solutions and explain solution processes. • Use patterns and relationships to analyze mathematical situations. • Believe that mathematics makes sense. • Recognize and apply deductive and inductive reasoning. • Make and evaluate mathematical conjectures and arguments. • Make and test conjectures. • Follow logical arguments. • Judge the validity of arguments. • Appreciate the pervasive use and power of reasoning as a part of mathematics.	*All students will...* • Draw logical conclusions from mathematical situations, using concrete models and verbal skills. • Understand and apply deductive, inductive and proportional reasoning, with special attention to spatial and visual reasoning with proportions and graphs. • Pose mathematical questions and evaluate arguments. • Validate individual thinking and intuition. • See how mathematics makes sense.

Connections

PreK–12 and Adult Basic Education

"All students should ...explore the relationship of mathematics to other areas of knowledge."

Massachusetts Common Core of Learning

Here is a model of the paradigm shift that summarizes mathematics education reform.

Moving away from...	Working toward...
...mathematics content that is broken down into several large strands, such as arithmetic and geometry. ...mathematics strands that are organized into a series of small, sequential objectives, which are taught separately, without exploring the commonalties of reasoning and skills in different strands. ...students learning new content by memorization of facts and formulas. ...students believing that mathematics is a collection of many isolated topics, rules, and procedures.	...mathematics content that is organized into separate standards, wherein students make connections among mathematical topics and domains. ...students making connections between mathematical ideas and other disciplines. ...students fostering meaningful connections between mathematical ideas and students' experiences in their daily lives.

For example...

... students connect their understanding of patterns to measurement when they make open boxes to explore the relationships of area to surface area or to volume, and surface area to volume.

Connections

Below are listed the standards that exemplify **mathematics as connections.**

PreK-12 Standards	Adult Basic Education Standards
All students will... • Link conceptual and procedural knowledge. • Relate various representations of concepts or procedures to one another. • Recognize and value the relationships among different topics in mathematics. • Use mathematics in other curriculum areas and in daily living. • Explore problems and describe results by using graphical, numerical, physical, algebraic, and verbal mathematical models or representations. • Apply mathematical thinking and modeling to solve problems that arise in other disciplines. • Recognize equivalent representations of the same concept. • Relate procedures in one representation to procedures in an equivalent representation.	*All students will...* • View mathematics as an integrated whole, which is connected to past learning, the real world, adult life skills, and work-related settings. • Explore problems by using appropriate technology, and describe results by using a variety of mathematical models and representations. • Apply mathematical thinking and modeling to solve problems that arise in other disciplines and in the real world, including work-related settings.

STRAND 1: NUMBER SENSE

Number sense is the cornerstone of mathematics. Sound number sense enables us to interpret and represent the world in which we live. It is exemplified every day, whether we consider notions as complex as the consumer price index, as pivotal as the impact of the Great Depression on United States history, or as personal as a blood pressure reading.

"None of us really understands what's going on with these numbers."

David Allen Stockman, Director USOMB, on the U.S. Budget, 1981

Youngsters' first use of numbers is with whole numbers. Soon, they use fractions when they share snacks with friends and decimals when they buy supplies to make valentines. As pre-adolescents, they explore negative numbers, perhaps with subzero temperature readings in Celsius. In high school, teenagers expand their understanding of real numbers when they collaborate to build a scale model of their remodeled school for a display at City Hall.

Sound number sense enhances mathematical intuition, which plays a significant role in mathematical problem solving. Mathematical intuition suggests appropriate strategies when confronting a problem situation.

Number Sense and Estimation

Number sense promotes accuracy in estimation and mental math. While calculators and computers are used to do most of the complex computations in today's world, the ability to estimate is critical for lifelong learners. Students estimate whether there is ample time to get to a movie before it begins. They estimate the cost of items they have gathered from recycling bins at the museum before they approach the cash register. Estimates are used to evaluate the reasonableness of a solution produced by a calculator —we all know the results of entering an extra zero or misplacing a decimal point.

Number Sense and Computation

Number sense suggests an ability to compute answers to problems involving quantities. Students figure out answers to problems long before they encounter formal mathematical representations. Younger students can quickly share evenly a dozen baseball cards among three friends before they have studied formal multiplication or division. Older students can find several ways to find the length of the side of a square with an area of five before they ever study irrational numbers.

Students should be taught how to do computational procedures. If they have developed their own accurate and sensible computational methods, then they should be encouraged to use them. A fourth grade student who finds the difference between 187 and 536 by mentally adding on 13 to make 200, adding 300 to make 500, and finally adding the 13 to 36 to get an answer of 349 exhibits a strong, internalized sense of number and a laudable proficiency with computation.

Number Systems and Number Theory

As students develop fluency with numbers and computation, they construct the scaffolding necessary for building an understanding of number systems and number theory. This may begin informally when second grade students investigate and explain what they find whenever they add an odd number to an even number. During elementary and middle school years, students extend their understanding from operations with positive numbers to those with negative numbers, and from rational numbers to irrational numbers. High school students examine in more depth the characteristics of the subsystems of the real number system.

Number Sense and Discrete Mathematics

Number sense includes the fuller, more accurate view and appreciation of mathematics that is experienced when students explore discontinuous or finite systems, also called discrete mathematics. Computer methods, compound interest, annuities, and arrangements in nature exemplifying Fibonacci sequences are everyday examples of discrete mathematics. Because many of the topics in discrete mathematics are accessible to students who do not have strong computational skills, it provides these students an accessible entry point into mathematics.

The nautilus shell is one of many examples of Fibonacci numbers that are found in the natural world.

Number Sense

What Is Discrete Math?

Look at all the things
That you can see...
And ask, "Can these
be counted
Individually?"

Ice cubes stack
Count 1, 2, 3.
Each cube sits alone
So discretely!

Water falls down
As a cascade from the top.
It is not so discrete
Unless you count each
drop.

Look for patterns
They are everywhere
On ceilings, floors, and
tables,
And in numbers that are
square!

Place your sharp pencil
On any point at the base.
Can you go over an
alphabet letter
Without having to retrace?

Sequence all the steps
Leading to a final result
This defines an algorithm
With numbers, or without.

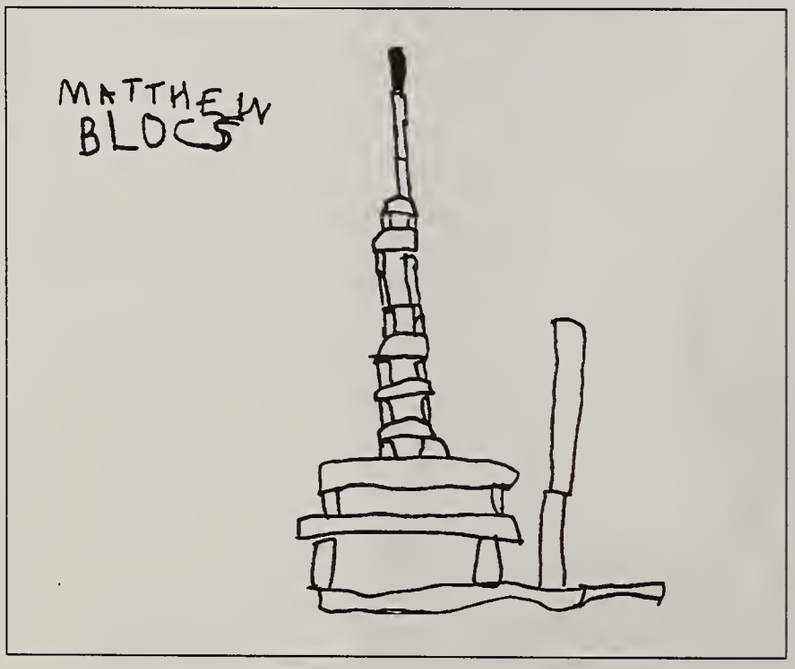
When completing these
examples
Your solutions may be
many.
Can you justify why your
answer
Is the best one of any?

Joan Fitton,
Worcester Public Schools

1. NUMBER SENSE
LEARNING STANDARDS FOR PreK – GRADE 4

1.1 Number Sense and Numeration

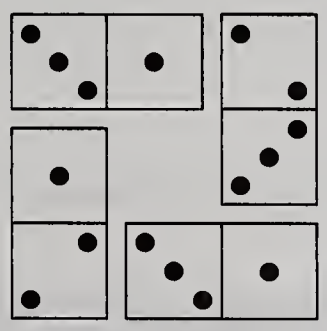
PreK–4 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • construct number meaning by using manipulatives and other physical materials to represent concepts of numbers in the real world. • demonstrate an understanding of our numeration system by relating counting, grouping, and place value concepts. • interpret the multiple uses of numbers by taking real-world situations and translating them into numerical statements.	• Students write an autobiographical number story wherein characters take on the identity of numbers. Number of siblings, grandparents, addresses, sports jerseys, Massachusetts zip codes, telephone area codes, and bus route numbers are encouraged. • When some classes go on field trips to such places as Old Sturbridge Village or Plimoth Plantation, the classes count themselves off as 1, 2, 1, 2, etc. before boarding the school bus. If you were the teacher, why would you use this counting method?



Matthew's drawing shows how he represents building height with blocks.

1. NUMBER SENSE
LEARNING STANDARDS FOR PreK – GRADE

1.2 Concepts of Whole Number Operations

PreK–4 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • model and discuss a variety of problem situations to help students move from the concrete to the abstract. • relate the mathematical language and symbolism of operations to problem situations. • identify a variety of problem structures that can be represented by a single operation. • know when to use the operations of addition, subtraction, multiplication, and division; and describe their relationships.	• Groups of students make board games that require players to use operations with whole numbers. Students could devise a game that requires them to add or subtract the numbers on dice. Others may use spinners or drawing cards to derive the numbers to be operated upon. Students name their games and write the rules of play. Groups of students then play each other's games, using manipulatives to help them. • Students work in pairs with four dominoes of the following descriptions: two are 1 dot and 3 dots; one is 2 dots and 1 dot; and one is 2 dots and 3 dots. They use the guess and check strategy to arrange them in a square so that sides each have the same number of dots.  • Students work in pairs to determine the number of outfits that a can be made for a snowman, using 2 hats and 2 scarves.

1.3 Fractions and Decimals

PreK–4 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • demonstrate an understanding of the basic concepts of fractions, mixed numbers, and decimals. • use models to relate fractions to decimals, find equivalent fractions, and explore operations on fractions and decimals. • apply fractions and decimals to problem situations.	• Students brainstorm as a whole group to write a letter to a sick classmate. Since the class has been investigating the relationship between fractions and decimals, members write about what they have learned this week in their mathematics class. Before the letter is mailed, each student draws a picture or writes a story that tells something about fractions and decimals, and they are included in the mailing. • Keep a monthly chart displaying students' birthdays. Make a grid divided into the appropriate number of days, for example five students, five blocks. Using two different colors, shade the appropriate number of blocks for boys' birthdays and the appropriate number of blocks for girls' birthdays. Express the findings in fractions. Have students use this method to record other findings of parts of wholes.

Using dominoes, these partners enjoy communicating about relating fractions to decimals.

1. NUMBER SENSE
LEARNING STANDARDS FOR PreK – GRADE 4

1.4 Estimation

PreK–4 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • describe the strategies used in exploring estimation. • determine when an estimate is appropriate. • apply estimation when working with quantities, measurement, and computation. • use estimation to check solutions to determine if the results of computational problems make sense.	• How many kernels of corn do students estimate are on an ear of corn? How did they arrive at their estimates? • Each student estimates the number of connecting cubes in a bucket by using a stack of ten cubes as a referent. They compare their estimates to the actual number by stacking the cubes in the bucket into towers of ten and counting them by 10s. Students are encouraged to revise their estimates as they stack towers. To extend the activity ask students to estimate the number of objects sold in containers, such as a box of apples. • Explain why you would estimate the number of plants needed at the base of the town's flagpole, but you would not estimate the number of teaspoons of fertilizer needed to keep the same plants healthy.

1.5 Whole Number Computation

PreK–4 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • model, explain, and develop proficiency with basic facts and algorithms. • use calculators in appropriate computational situations. *Second graders show what they know and are able to do by creating, then displaying, t-shirts.*	• A play store is set up in the math corner. Empty boxes, cartons, cans, and plastic fruit (if available) are tagged with price tags. Play money is provided in a home-made cash drawer. Groups of students play store. A pair of students works together as the store owners, and others play the role of shoppers. Roles change, to allow all students the opportunity to compute and make change in the context of grocery shopping. • Give each pair of students a set of dice and a recording sheet with ten-spaced columns numbered 1–12. Have students take turns rolling the dice and recording the addition fact rolled in the appropriate column. • *The Doorbell Rang* by Pat Hutchins (available in English and Spanish editions) is a story of visitors arriving consecutively, and then sharing cookies with each other. It provides opportunities for students to develop and share computation strategies. In a first grade classroom, groups of students each have a dozen paper cookies and share them fairly, pasting an equal number on paper plates then writing a number sentence to describe what they did. In a third grade classroom, students arrange cookies in arrays on baking sheets and consider whether they have enough cookies for their guests. Students use calculators to check their work. • Students generate multiples of 9, using paper and pencil. They discuss the number patterns that they see. The teacher guides students, to help them see that adding the digits of each multiple results in the sum of 9. Students use calculators to generate larger and larger multiples of 9 and mentally add the digits to see that the pattern of the sum as a multiple of 9 continues. As an extension, students explore multiples of other numbers. • Students predict shortest and longest routes, using a map. They use calculators to check their predictions.

How It Looks in the Classroom

In a Kindergarten class, Ms. Chapman uses autumnal changes as an interdisciplinary theme. Students talk about the seasons and observe how the change from summer to winter affect plants and animals. They listen to stories about autumn, and draw pictures of what they notice when the seasons change. During a walk on school grounds, each student picks a leaf to bring back to the classroom. The class is seated on a rug, alongside their pile of leaves.

"What can we say about our pile of leaves?" she asks.

Soleap says, "There are lots of them."

"How many do you think there are?" asks Ms. Chapman.

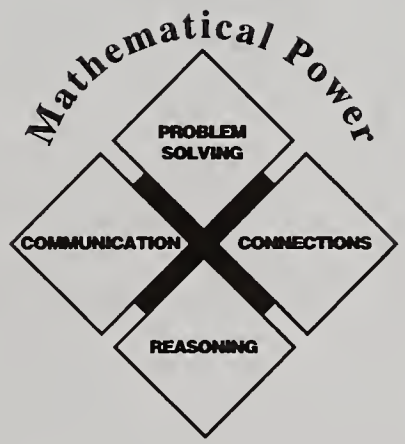
"More than 100," Rachel estimates. Actually, there were 23. Ms. Chapman is trying to build the students' sense of quantity, but realizes that to several students, 100 means "a lot." She is sensitive to Rachel's need to develop the concept of one-to-one correspondence.

"What else do you notice about the leaves?" asks Ms. Chapman.

"They are different colors," says Pat.

"Some are yellow, and some are green, and some are red, and some are brown, and some are mixed-up," said Floyd.

Ms. Chapman suggests sorting the leaves into piles by color, which involves frequent discussions about into which pile a multi-colored leaf fits. Ms. Chapman wants students to realize that identifying attributes and categories can be ambiguous—defining the criteria for a particular group is sometimes arbitrary.



"Look at the pile of green leaves and the pile of yellow leaves. What can you say about them?" Ms. Chapman wants them to compare quantities and use the terms more than and less than.

"Some are big and some are small," says Leah.

"Which leaf do you think is the biggest?" Ms. Chapman asks. Discussing which leaves are big and which are little is also ambiguous since some leaves are longer than others, while other leaves are wider, and still others seem to be larger in terms of area. This experience wherein students informally measure by comparing the lengths or widths of leaves helps lead to an intuitive sense of measurement.

"Can we put all the leaves in order in a long line so the smallest leaf is at the end?" This sequencing gives students the opportunity to see increasing/ decreasing relationships. Ms. Chapman observes that while some students try to put the leaves in order by using random techniques, others begin to compare each leaf as it is added to the series. "Let's see how many different ways we can sort the leaves, putting them into groups," suggests Ms. Chapman.

1.6 Number and Number Relationships

Grades 5–8 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • represent and use equivalent forms of numbers, including integers, fractions, decimals, percents, exponents, and scientific notation. • apply ratios, proportions, and percents. • investigate and describe the relationships among fractions, decimals, and percents. • represent numerical relationships in one- and two-dimensional graphs.	• Students use a spreadsheet to work with ratios as they consider recipes for small and large groups. • As a fundraiser to defray the cost of paint, City Year participants are selling pizzas at one of their city-wide beautification days. Six-inch individual cheese pizzas were $2.75 last year and will be the same this year. What is a fair price to ask for their new 12" family cheese pizza? Students justify their answers. • Students go to the supermarket to record prices of different size containers of identical products. Later their data are used to determine the best buy. As students justify their reasoning, using ratios and comparisons, extraneous factors such as food spoilage and storage are considered. • How many times greater is your height than the circumference of your head? Students predict, then measure a sample to determine the ratio and check their predictions. What other human body ratios can students think of to predict and check? Students state their hypotheses and test them.

1. NUMBER SENSE
LEARNING STANDARDS FOR GRADES 5 – 8

1.7 Number Systems and Number Theory

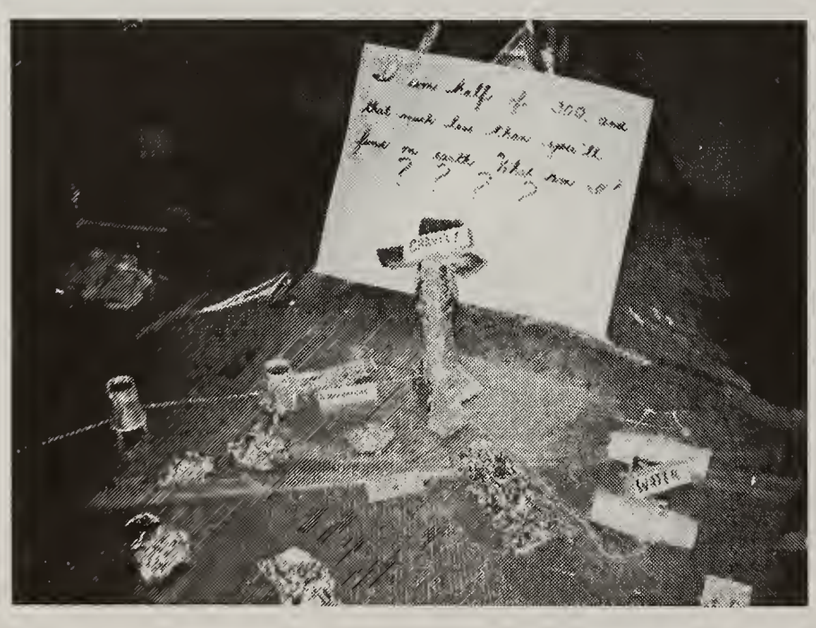
Grades 5–8 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • explain the need for numbers other than whole numbers. • know and use order relations for whole numbers, fractions, decimals, integers, and rational numbers. • use operations involving fractions, decimals, integers, and rational numbers. • demonstrate how basic operations are related to one another. • create and apply number theory concepts, including prime numbers, factors, and multiples.	• Groups of students chart temperature and wind chill over the winter months. They find and compare the mean and the median, and consider which would be more useful to meteorologists. • Students choose a company's stock to follow in the newspaper. They note changes and convert the reported fractions to decimals in order to compute the monetary gains or losses.

Each member of this group is actively engaged in finding the missing piece.

1. NUMBER SENSE
LEARNING STANDARDS FOR GRADES 5 – 8

1.8 Computation and Estimation

Grades 5–8 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • compute with whole numbers, fractions, decimals, integers, and rational numbers. • develop, analyze, and explain procedures for computing, estimating, and solving proportions. • select and use an appropriate method for computing from among mental arithmetic, paper-and-pencil, calculator, and computer methods. • use computation, estimation, and proportions to solve problems. • estimate to check the reasonableness of results of computations and problems involving rational numbers.	• Students work in groups to estimate the amount of water wasted and its cost to each tenant in an apartment complex, when a leaking water faucet goes unrepaired. They report their findings, as well as the strategies they used for collecting and analyzing their data. Each group conducts research to determine the most cost-effective method of repair. Students write to the Education Department of the Massachusetts Water Resources Authority, 100 First Avenue, Building 39-4, Charlestown, MA 02129 for literature discussing water conservation.



I am half of 300, and that much less than you'll find on earth. What am I?

A student's work challenges other students to use logical reasoning combined with estimation skills to help solve this riddle.

1. NUMBER SENSE
LEARNING STANDARDS FOR GRADES 5 – 8

How It Looks in the Classroom

Ms. Cho: If we know from our reading that the area of the square for a tern nesting site is 2 square feet, then how can we find the length of each side to use as a reference point for fencing a larger area to preserve nesting sites at the Cape Cod National Seashore?

Sandy: Find a number that when multiplied by itself equals 2.

Ms. Cho: Tom suggested 1.5. Use your calculators to see what 1.5 squared is.

Ms. Cho: Did 1.5 x 1.5 equal 2?

Stan: No, it was 2.25. That's too big.

Ms. Cho: What's another estimate for the length of the side?

Leon: I think 1.25 . . . that's between 1 and 1.5.

Judy: Yes, but 2.25 is only a quarter away from 2, and 1 is a whole unit away from 2, so the answer must be closer to 1.5 than to 1.

Felipe: I'm going to try 1.3.

Ms. Cho: OK, let's try both those numbers. (pause) What did you get?

Leon: 1.25 times 1.25 is 1.5625. That's too small.

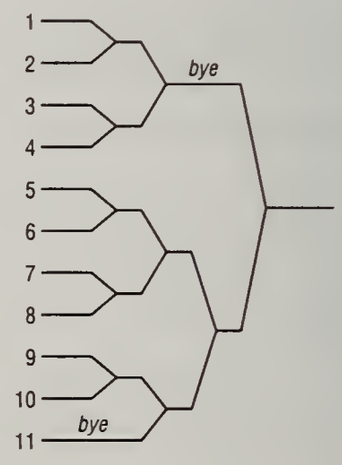
Felipe: 1.3 times 1.3 is 1.69. 1.3 is too small too.

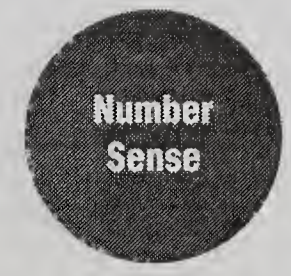
Joe: 1.4 times 1.4 is 1.96. That is real close. It's going to be just a little more than 1.4. Maybe it's 1.5.

Susan: No! That's what we tried first!

Ms. Cho: Let's see what the rest of you think. In the next few minutes, see if you can get closer to the number 2 by squaring a number. You may choose to work by yourself, with a partner, or with your group of four. Keep a list of the numbers you try and their squares. Then we'll share what we find out.

1.9 Discrete Mathematics

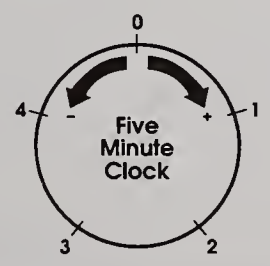
Grades 9–10 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • represent problem situations, using discrete structures such as finite graphs, matrices, sequences, and recurrence relations. • represent and analyze finite graphs, using matrices.	• Students write an autobiographical number story. • Students calculate the number of "byes" necessary for a tournament involving many teams or players. Here is an example for 11 teams. •  Students use graphs and matrices to represent and analyze many types of networks. One group decides to analyze the routes involved in getting from their homes to various places in town. Using a map of their town, they label intersections and note whether streets are one- or two-way. They arrange the information in matrices and use graphing calculators to determine the number of routes from one point to another. *Continued on next page*



1.9 Discrete Mathematics, continued

Grades 9–10 Learning Standards	Example of Student Learning
Continued from previous page	• Students use the sequence, table, and list features of their graphing calculators (or spreadsheet programs) to discover and explore the Fibonacci sequence and its relationship to other sequences and ratios also found in many natural and man-made structures. (*Mathematics Teacher*, Vol. 88, No. 2, February 1995, pp. 101-105.)

1.10 Mathematical Structure

Grades 9–10 Learning Standards	Example of Student Learning
Students engage in problem solving, communication, reasoning and connections to: • compare and contrast the real number system and its subsystems with regard to structural characteristics. • demonstrate the logic of algebraic procedures and their interrelationship with geometric ideas and concepts.	• Students determine patterns for quilts by using clock arithmetic (see below)to complete fact tables. They assign a color to each number and color their charts, which become the pattern for their quilt patches. They rotate and reflect their patch to get different pattern combinations for a quilt. 

1.11 Estimation

Grades 9–10 Learning Standards	Example of Student Learning
Students engage in problem solving, communication, reasoning and connections to: • use estimation strategies to judge the reasonableness of results of computation and problem solving involving real numbers. • use estimation when making graphs.	• When Benjamin Franklin died in 1790, his will stated that $5000 be given to Boston to be used for public works. Part was to be used after 100 years and the rest was to be used after 200 years. Estimate the amount of money that could be available now, if the funds were invested at 5% compound annual interest. List your assumptions and justify your answer in terms of your assumptions. Explain the differences or similarities between your estimate and that of one of your classmates.

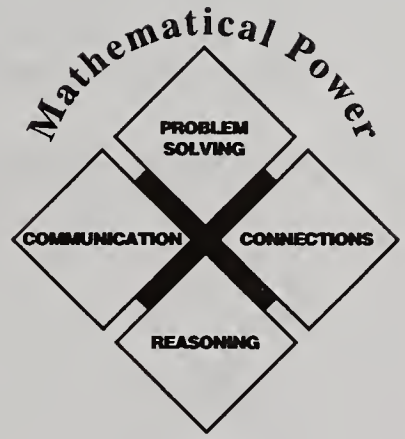
1. NUMBER SENSE
LEARNING STANDARDS FOR GRADES 9 – 10

How It Looks in the Classroom

Groups of students have several large drawings of polygons: quadrilaterals, pentagons, hexagons, septagons and octagons. After estimating the sum of the measures of the interior angles of each of the polygons, they measure each angle and try to come to agreement within the group. Students make charts showing their results. They predict the result for an n-gon and develop a formula. Using a graphing calculator, students enter the sequence and determine a formula.

Name of Polygon	Number of Sides	Sum of Angles
triangle	3	180°
quadrilateral		
pentagon		
hexagon		
septagon		
octagon		

Students then look at the problem in another way, by drawing in all the diagonals that can be made from one vertex of the polygon. This allows them to extend their chart and confirm their predictions.



Name of Polygon	Number of Sides	Sum of Angles	Number of Triangles	Sum of Angles as multiple of 180
triangle	3	180°	1	1 x 180
quadrilateral	4	360°	2	2 x 180
pentagon				
hexagon				
septagon				
octagon				

Extend the activity to a study of three-dimensional figures and crystals. Or, study the sums of the measures of the exterior angles of polygons.

Students with access to computers and software for learning geometry can use them for these explorations.

1. NUMBER SENSE
LEARNING STANDARDS FOR GRADES 11 – 12

1.12 Discrete Mathematics

Grades 11–12 Learning Standards	Examples of Student Learning
Students engage in problem solving, communication, reasoning and connections to: • use inductive and deductive reasoning to investigate problem situations involving networks. • represent and solve problems by using linear programming. • invent and analyze algorithms. • use the algebra of matrices to solve problems involving finite graphs. • investigate and describe problem situations that arise in connection with computer and calculator validation and the application of algorithms. • solve enumeration and finite probability problems.	• Students find the location of Steiner points for three cities or towns in Massachusetts. They locate the points experimentally, using geometry computer programs or coordinate geometry and trigonometry with calculators and/or computers. As an extension, students may find and use geometric constructions to locate points, and use deductive reasoning to validate their findings. • On a history field trip to the Museum of Fine Arts, some mathematics students visit the Nubian Gallery and look at the burial ornaments in the collection. In their science and mathematics classes, they explore carbon dating and find the approximate ages of the ornaments they have seen. • Stick, Flip, or Switch? A game show host tells you that behind one of three doors is a fabulous prize. The other two doors lead to a booby prize. You choose a door. After you choose, the host, who knows where the prize is, always opens one of the remaining doors and reveals a booby prize. At this stage of the game, you are allowed to switch the door you selected to the other unopened door if you wish, before the prize is revealed. What is your best strategy, stick to your original selection . . . flip a coin to decide . . . always switch?[4]

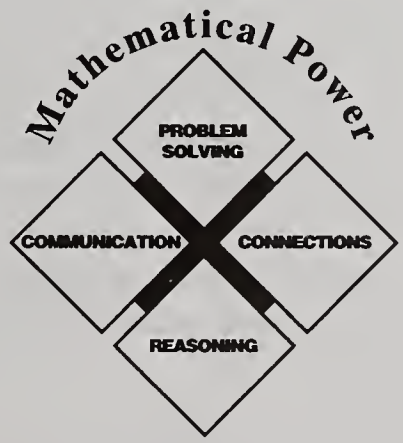
1. NUMBER SENSE
LEARNING STANDARDS FOR GRADES 11 – 12

1.13 Mathematical Structure

Grades 11–12 Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to:	
• explain why seemingly different mathematical systems may be essentially the same.	• Students are guided toward finding the connections among designing electrical circuits (series/parallel), searching an electronic database (and/or), and using binary operations (multiplication/addition).

How It Looks in the Classroom

Ms. Zangari introduces the "Shortest Network" topic to her class by presenting them with a problem that was generated in 1960 when a national airline company claimed that a major telephone company was over charging them for the private lines between their main terminals in three major cities. The mathematicians at the telephone company labs researched the claims. Based on the theory developed in the 1800s by German mathematician Jacob Steiner, the lab decided that the airline was correct. A fourth point could be located so that the shortest route between the cities is a minimum, and this route could be the basis for the phone line charges. Today, the telephone companies are not really concerned with the location of Steiner points, but in the length of the network that results if Steiner points are used. The cost of the service is based on the total length of the network.



In the first part of the investigation, students use rulers and protractors to determine the shortest network of line segments that will interconnect three given points. Presentations of findings and guided discussion leads the students to recognize that the Steiner point will be located so that the angles formed by the segments connecting the point to each of the original points all have the same measure. The students make templates on overhead transparencies that can later be used to locate the point.

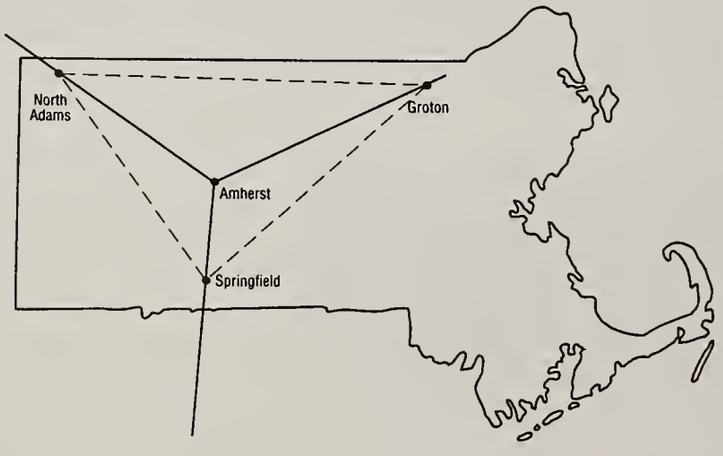
How It Looks in the Classroom, continued

The second part of the investigation has students using their "Y" transparencies to locate actual Steiner points, first for the three cities involved in the airline problem, then for any three cities in the United States, and finally here in Massachusetts.

The third (with some classes, this is first or second) part of the investigation has students using Plexiglas models of the three cities and a soap bubble solution. When the model is dipped into a soap bubble solution, and placed on an overhead projector, the network joining the three points is clear. Students use their "Y" transparencies to validate that a Steiner point has been located.

The investigation continues, with students exploring models with more than three vertices and discovering that there may be more than one Steiner point. Students develop an algorithm to cluster subsets of three points, then form Steiner points in these subsets and measure the networks. Students derive formulas for the number of possible Steiner points for a three-point connection, a four-point connection, a five-point connection, and so on. Some students may extend this to write a procedure to create Steiner points for various geometric computer programs.

This investigation can be modified for use at any grade level. At the elementary level, it is used for pattern recognition, measurement, and the interdisciplinary purpose of the science of soap bubbles and geography with map reading. In middle grades, teachers use the investigation for skill development in measurement of segments and angles. The hands-on approach lends itself to numerous open-ended discussions. Secondary teachers use all of these approaches as an introduction, then proceed to prove the Steiner point theorem and solve real-world problems through scenarios involving connection of points as applied to analytic geometry and calculus.



Amherst is the Steiner point (junction point) that forms the shortest network connecting North Adams, Groton, and Springfield, Massachusetts

1.14 Estimation

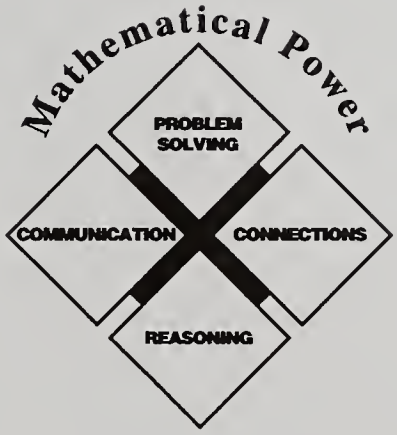
ABE Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • explain the strategies used to explore and develop estimation techniques with whole numbers, fractions, decimals, and percents. • identify when an estimate is appropriate. • apply estimation techniques to quantity, measurement, computation, problem solving, workplace, and test situations. • estimate to check the reasonableness of results.	• Small groups of students share strategies for figuring out the approximate meals tax and tip at a restaurant, without using a pencil or calculator. Part of the discussion involves ways to determine an appropriate percentage for the tip. Later, the whole class compiles one list.

1.15 Number, Number Relationships, Operations, and Computation

ABE Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • represent and use numbers in a variety of equivalent forms and in order relations in real-world, work-related, and mathematical problem situations. • compute with whole numbers, fractions, decimals, and integers, using appropriate algorithms and a variety of techniques, including mental math, paper-and-pencil, calculator, and computer methods. • analyze and explain procedures for computation and how operations are related to one another, particularly the reversibility of operations. • estimate to develop number sense, operation sense, and to check the reasonableness of results. • analyze and explain methods for solving proportions. • use computation and proportions to solve problems. • select and use, from among mental arithmetic, paper-and pencil, calculator, and computer application, an appropriate method in problem-solving situations.	• Students go to local stores to determine the prices of different brands of a household item such as toothpaste. They decide which is the best buy for them and explain their reasons. Their results may not necessarily be the least expensive toothpaste. They may bring up other factors such as size, taste, or fluoride. They may prefer a more expensive smaller tube over a cheaper larger tube because the larger tube tends to get messier and more toothpaste gets wasted. Unit pricing is an important piece of this activity, but students should see that in real life, the lowest unit price is one of several factors to consider when making a purchase. • Students work in groups to compare different cultures' procedures for division. Each member compares and contrasts one procedure with another, and explains why each method works.

1. NUMBER SENSE
LEARNING STANDARDS FOR ADULT BASIC EDUCATION

How It Looks in the Classroom



One teacher asked a group of General Education Development (GED) mathematics students to tell her how much it would cost to buy four shirts for $7.98 each. She told the students to figure it out any way they wanted, except that they could not use pencil and paper. She noticed that they used their fingers in the air or "wrote" on their desks. Most students were able to multiply and get the right answer. When asked how they got their answer, the students responded that they multiplied $7.98 by 4.

The teacher then asked the class what they would do if they were in a store, and needed to determine the total cost for the four shirts. All agreed that they probably would not have solved the problem the same way. Some thought they would have multiplied 4 x 7, plus 4 x 1, and then subtracted 8 cents from that total. Others said they would round $7.98 to $8.00, multiply that by 4, and then subtract $0.08 from the product.

Finally, the teacher asked why no one solved the problem either of these latter ways when given the problem in the classroom, to which students responded, "This is math class, so we need to work out the problems."

STRAND 2:
PATTERNS, RELATIONS, AND FUNCTIONS

"Mathematics is an exploratory science that seeks to understand every kind of pattern—patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns. To grow mathematically, children must be exposed to a rich variety of patterns appropriate to their own lives through which they can see variety, regularity, and interconnections."

Lynn Arthur Steen, On the Shoulders of Giants

Patterns, Relations, and Functions form the base of mathematics. Recognizing patterns and describing their relations mathematically by using geometry, number sequences, and functions, helps us interact with and make sense of our world. To appreciate fully the intrinsic value of such pleasures as poetry, art, music, plants, and animals, lifelong learners should know the mathematics of patterns and use mathematical representations to describe them. The terminology of patterns, relations, and functions has become a part of our culture. Headlines and news reports speak of exponential growth of the national debt, a variable rate mortgage, or a balanced budget.

Students discover patterns on leaves and stems as they observe plants growing at the science table. They marvel at the patterns on fish they see in aquariums. Quilt patterns made with fabric or on dot paper are other examples. These real-world examples of patterns serve as segues to numerical and geometrical investigations of mathematical patterns and functions, such as skip-counting, odd and even numbers, square numbers, triangular numbers, Pascal's triangle, and the Fibonacci sequence.

Concepts for algebra are first formulated in Pre-K. By the end of eighth grade, algebraic concepts should be well-developed. Algebra learning continues through adulthood. Furthermore, it not only serves as a connector of the topics in mathematics, it is also recommended as a venue for interdisciplinary curriculum.

Patterns and Relations

Younger students identify, translate, and extend repeating rhythmic, verbal, and visual patterns. They recognize patterns and relationships among objects when they sort and classify them, identifying similarities and differences. Groups of students may make a list of insects and the corresponding sequential total number of legs, or fellow students and the number of shoes, as they begin developing their own understanding of functions and multiples.

Patterns and Functions

As students develop, their work with patterns and functions moves concurrently toward mathematical sophistication. They identify and represent numerical relationships in tables, charts, and graphs. Perhaps students work in groups to plan and map a trip from their home to either coast, showing the relationship between constant speed and distance with a linear graph. They may also collect time and distance data of a local trip that show how average speed represents only one aspect of the trip, displaying these real data on a non-linear graph. Software programs featuring spreadsheets and graphing capabilities are ideal resources for teachers and students.

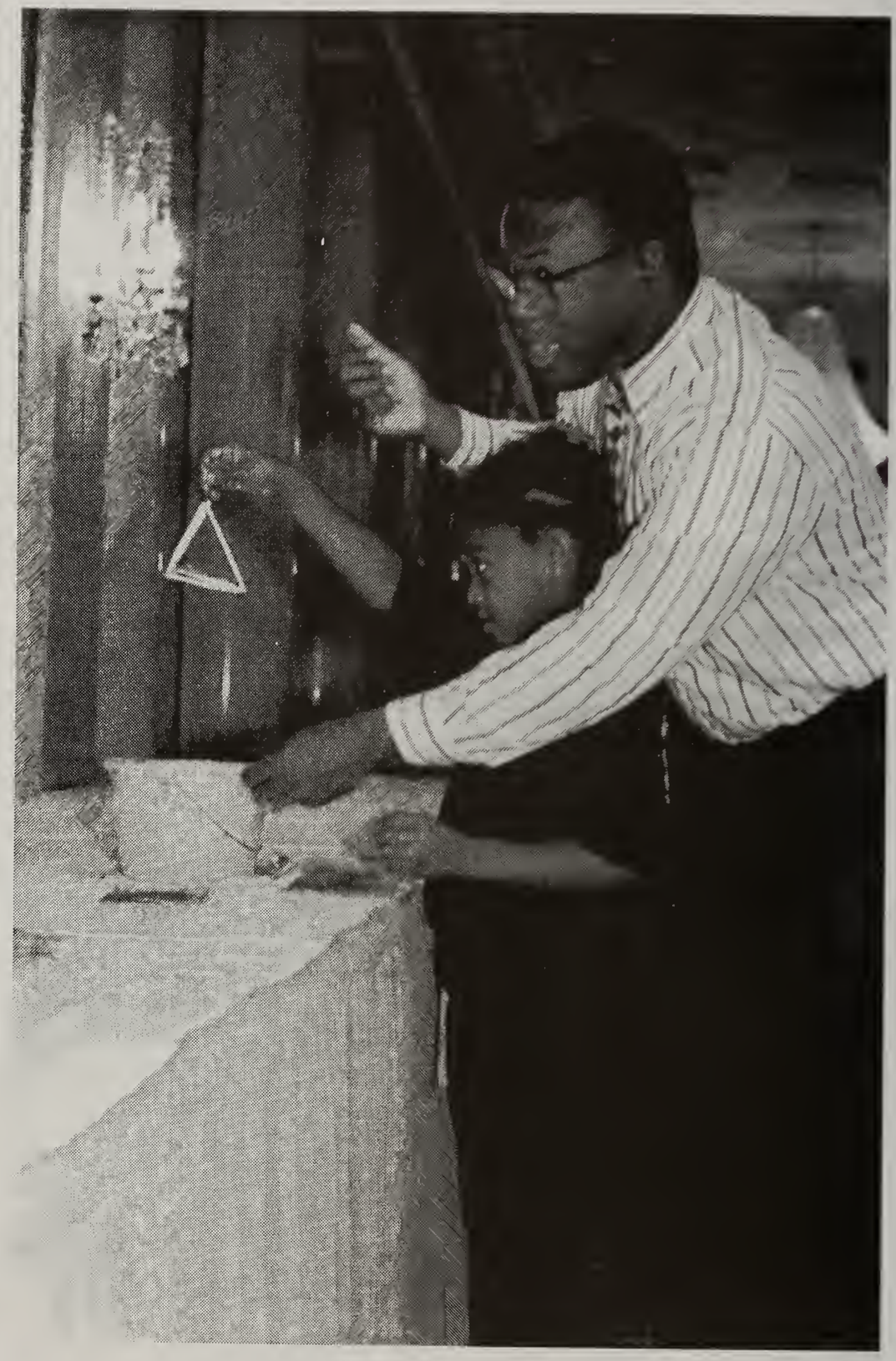
Beyond Patterns and Functions

Patterns and functions are the building blocks for transformational geometry, algebra, discrete mathematics, trigonometry, and calculus. Underpinnings of calculus can be seen in algebra and geometry during earlier grades. Older students will explore ideas about continuity, discontinuity, maximum and minimum. Real world optimization problems, such as those related to maximum profit earned or minimum height achieved, provide an opportunity to see the relevance of these concepts. Technology such as mathematical software and graphing calculators can be used to build understanding of function, explaining their domain and range, and comparing and translating among various representations.

Patterns, Relations, and Functions in the Workplace

Life experience offers adult basic education learners a broad base of real-world ties that are readily linked to the concepts of equation, function, variable, and graph. Each of these are representations of patterns, relations, and functions. From baby formulas to chemical formulas, algebra offers a succinct way to define real-world situations that can aid adults in the home and in the workplace.

These elementary students are exploring algebraic concepts by manipulating tiles to form patterns.



By using inquiry as a means of exploring bubbles and geometric shapes, this student discovers surface tension and shortest network.

2. PATTERNS, RELATIONS, AND FUNCTIONS LEARNING STANDARDS FOR PreK – GRADE 4

2.1 Patterns and Relationships

PreK–4 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • identify, describe, extend, and create a wide variety of patterns. • represent and describe mathematical relationships. • explore the use of variables and open sentences to express relationships. • use patterns and relationships to analyze mathematical situations.	• Make a group chart, counting the number of eyes in the class. Give students some type of manipulative for them to use when representing eyes. Fill in the pattern on a 0–99 chart. Discuss the numerical pattern. Use the constant key on the calculator to extend the pattern. • Introduce Fibonacci number patterns. Beforehand, the teacher uses different colors of poster paint on pine cones to identify spirals. Small working groups count the numbers of bracts of the same colors that form the spirals, and record their data on charts. Some spirals are steep, others are gradual. What they discover is an example of the Fibonacci number sequence. The activity is extended when students cut spirals from colored squares and hang them from the ceiling. • Students look at and discuss patterns and symbolism of the United States flag and those of other countries. They design a mathematics flag for the classroom. They may make it any size they wish, choosing from an array of art materials. The only requirement is that it shows a pattern that is repeated. Transformations (flips, slides, and turns) may be encouraged. Pairs of students identify and describe each other's patterns.

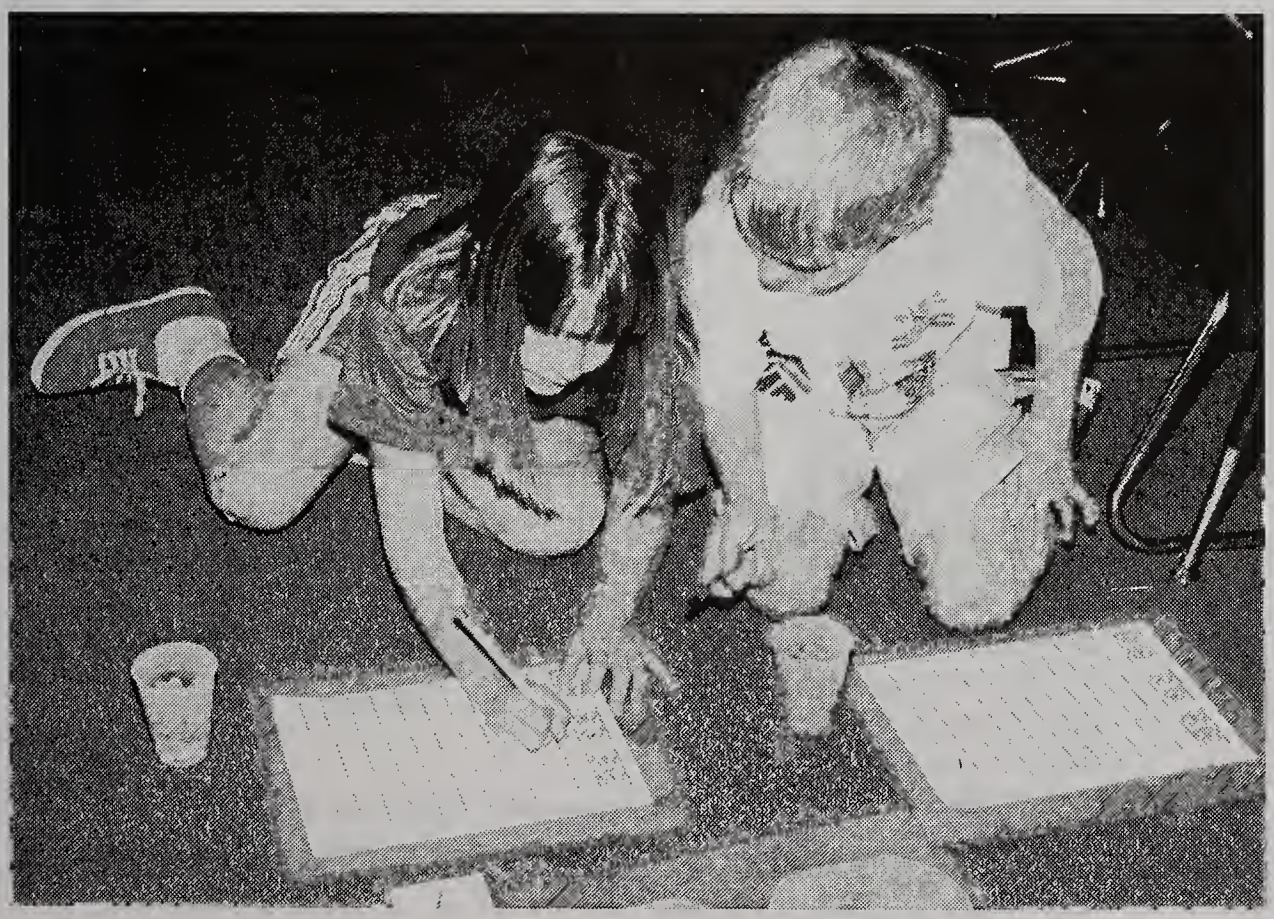
2.2 Algebra

PreK–4 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • discover how to form, then write, number sentences for real problems. • investigate and describe ways to find missing components in number sentences. • demonstrate through hands-on activities, an understanding of maintaining balances in number sentences. • explain the use of variables in number sentences.	• If you were to make cheese sandwiches for yourself and a friend, then how many more pieces of bread would you need than cheese? Is there a mathematical way to write this? If so, what is it? • Students use shoe boxes to make an input/output mathematics machines, which they decorate. Using counters as tokens, they can explore fact families, properties, functions, and computational skills. When students are ready to move to a more abstract model, they can use a function machine that is represented on paper to play Guess the Rule.

2. PATTERNS, RELATIONS, AND FUNCTIONS LEARNING STANDARDS FOR PREK – GRADE 4

2.3 Mathematical Structures

PreK–4 Learning Standard	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • explore and demonstrate an understanding of commutative properties for addition and multiplication.	• Students think of commutative and non-commutative tasks. For example, does it matter whether I brush my hair first or my teeth first? Does it matter whether I brush my teeth first or eat my chocolate creme-filled cookie first? • Students explore commutative in the sense of commuting to work or school. Does it matter which way you go when you determine the distance of the commute?



Collaborating to solve a problem encourages communication of mathematical ideas and prepares learners for the workplace.

2.4 Patterns and Functions

Grades 5–8 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • describe, extend, analyze, and create a wide variety of patterns. • describe and represent relationships with models, tables, graphs, and rules, using sentences and algebraic expressions. • analyze functional relationships to explain how a change in one quantity results in a change in another. • use patterns and functions to represent and solve problems. *Manipulatives come in all kinds of packages! Here students use sugar cubes as materials for building scale models of ancient architecture.*	• Students use color interlocking cubes to build a sequence of towers, pyramids, or other patterns. As each sequence is built, students describe the pattern they see and predict, then build, the next two shapes. They organize their information by making a table, describing the pattern in writing, writing a rule, and plotting points on a coordinate graph. • Reduce, Reuse, Recycle. What does this mean in your classroom and at your school? Could more be done to encourage recycling? Small groups of students gather data that explain the present recycling practices in their classroom and support their suggestions for improvement. They work together to write a proposal for garbage and trash reduction. It includes graphs, tables and charts as well as cost analysis. Students make posters for display and present their findings to the community and administration.

2. PATTERNS, RELATIONS, AND FUNCTIONS LEARNING STANDARDS FOR GRADES 5 – 8

2.5 Algebra

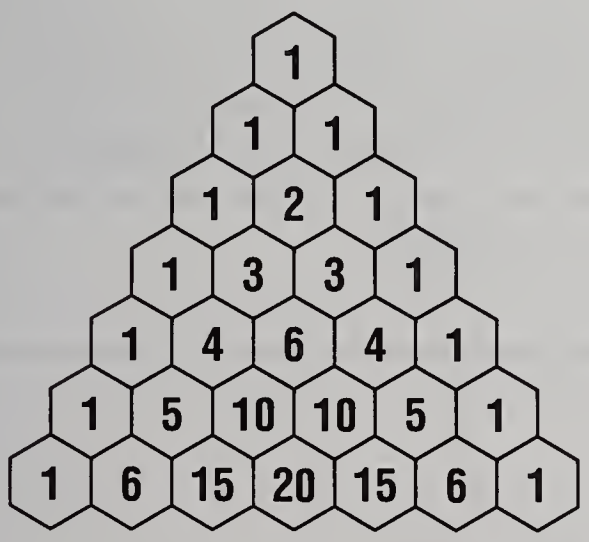
Grades 5–8 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • understand and apply the concepts of variable, expression, and equation. • represent situations and number patterns with tables, graphs, verbal rules, and equations and explore the interrelationships of these representations. • analyze tables and graphs to identify properties and relationships. • demonstrate an ability to solve linear equations, using concrete, informal, and formal methods. • describe the strategies used to explore inequalities and nonlinear equations. • apply algebraic methods to solve a variety of real-world and theoretical problems. • construct expressions or equations that model problems. • explore and describe a variety of ways to solve equations, including hands-on activities, trial and error, and numerical analysis. • know and apply algebraic procedures for solving equations and inequalities.	• The snowy tree cricket sings a distinctive song on summer nights. Scientists have discovered that when 37 is added to the number of chirps heard in 15 seconds, the current Fahrenheit temperature is known. Students make tables and graphs, then write an equation and a rule for this phenomenon. They conduct library research or observe nature to discover another such phenomenon, then write about it. • Students begin by working in groups and dropping a ball from a specified height. They observe how high it rebounds on the first bounce. They find the average first rebound from the given height and predict the height of the second rebound. They test their predictions. Students develop a model for the situation and predict the height after 4, 5, 25, 100, 500 falls. They compare the elasticity of similar and different balls. How much variation is there among tennis balls from the same can? How much variation is there among different brands of tennis balls? How do tennis balls compare with basketballs and baseballs?

2.6 Algebra

Grades 9–10 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • formulate problems that involve variable quantities with expressions, equations, and inequalities. • use tables and graphs as tools to interpret expressions, equations, and inequalities. • simplify algebraic expressions to solve equations and inequalities.	• One "Ask Marilyn" feature in Parade magazine read: *Half the employees in a firm went to lunch at noon. Since then, 25 have returned and 7 others have gone out. At this point, there are twice as many people working as there are people out to lunch. How many people are employed by the firm?* Students explore this problem in groups, comparing solutions and various methods for determining them. They write equations and explanations describing their solutions. Groups of students make up a similar problem, then solve each other's problems. • The 807 airliner holds a maximum of 200 passengers. A carrier has a price structure that states if the airline is full, each passenger pays a $200 fare. For each drop of 5 passengers, the price per passenger increases by $10. Find the price that will maximize the income of the airline.

2.7 Functions

Grades 9–10 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • model real-world phenomena with a variety of functions. • represent and analyze relationships, using tables, verbal rules, equations, and graphs. • translate among tabular, symbolic, and graphical representations of functions.	• *Algebra for the Twenty-first Century: Proceedings of the August 1992 Conference* presents this problem: A patient is taking approximately 16 mL of medicine every four hours for a long period of time. How much of the medicine will eventually be in her blood? Assume the body eliminates 25% of the medicine in every four-hour interval. Students can investigate this problem in several different ways. They may carry out a simulation, set up a table by using arithmetic calculations, use a spreadsheet, apply the formula for the sum of a geometric series, or use the recursive mode on their graphing calculator. • Students explore and discuss the patterns and exponential representations in Pascal's triangle. They devise a rule for finding the sum of the nth row in the triangle. Using what they have learned, they formulate and solve their own probability questions that can be answered by using Pascal's triangle.



Pascal's triangle

2.8 Trigonometry

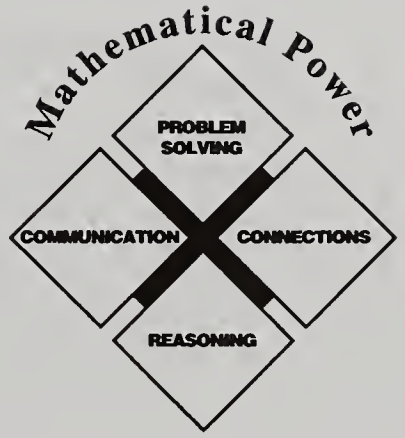
Grades 9–10 Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to:	
• apply trigonometry to problem situations involving right triangles.	• Students replicate the experiment in which Eratosthenes calculated the circumference of the Earth and got a remarkably good answer. They locate some schools roughly due north or south and connect with them through electronic mail. Students in each school agree that on a given day, at high noon, they will measure the shadow cast by a vertical stick on level ground. After sharing the measurements of the stick and the shadow, students use trigonometric ratios to determine the angle of the sun's rays. Using this information along with approximate distances between schools, the students use proportions to find an approximation of the Earth's circumference. This example can be extended to sharing data with students from other states and countries.

2. PATTERNS, RELATIONS, AND FUNCTIONS LEARNING STANDARDS FOR GRADES 9 – 10

How It Looks in the Classroom

Ms. Arroya has students bring in examples (from mail, newspaper, TV, or other advertisements) involving rates. Students work in small groups to analyze them. Here are some samples.

Recently, a bank in Boston advertised a special for customers who had a home equity account with them. The offer was stated, "Borrow at least $2500 by April 30, and receive a certificate which can be used for up to $25 in long distance telephone calls with a specified phone service." Is this a good deal?



Two banks in the Worcester area recently ran competing ads for their savings accounts. One offered a rate of 6% compounded monthly, while the other offered a rate of 5.75%, but compounded daily. If you had $100 to invest, what would you do? What are some of the factors that might affect your decision?

Mario did not bring in an ad, but he posed a problem from something his mother told him about interest. She said there is a "Rule of 72," which holds that the number of years it takes an investment to double in value when the interest is compounded can be approximated by dividing 72 by the rate (multiplied by 100). For example, if an amount is invested at 8%, it would take about 9 years to double in value. Does this rule work? The class explored the rule and decided its usefulness by trying to prove it.

2. PATTERNS, RELATIONS, AND FUNCTIONS LEARNING STANDARDS FOR GRADES 11 – 12

2.9 Algebra

Grades 11–12 Learning Standards	Example of Student Learning
Students engage in problem solving,communicating, reasoning, and connecting to: • represent situations that involve variable quantities with matrices. • use tables and graphs as tools to interpret higher order equations, inequalities, and matrices. • use matrices to solve linear systems.	• Students can use the matrix feature of graphing calculators or computer programs to solve systems of linear equations generated by real problems. They may use information gathered from medical and law enforcement authorities to solve problems connected with amounts of toxins in a person's system. They use graphing calculators or computer programs to create lists, graph functions, and to investigate how changes in one variable will affect others.

2.10 Functions

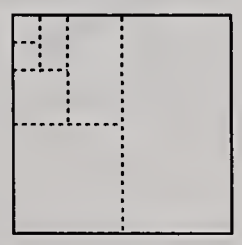
Grades 11–12 Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • model real-world phenomena with a variety of nonlinear functions. • identify a variety of problem situations that can be modeled by the same type of function. • analyze the effects of parameter changes on the graphs of functions.	• Students use parametric equations to solve problems they have posed based on movement over time. One group of students may suggest the problem of ships passing (or colliding) in the night. Using graphing calculators and parametric equations, the students can simulate the actual passage of the ships, and can determine visually, as well as algebraically, how far apart they are at any given time and when they pass (or collide).

2. PATTERNS, RELATIONS, AND FUNCTIONS LEARNING STANDARDS FOR GRADES 11 – 12

2.11 Trigonometry

Grades 11–12 Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • apply trigonometry to problem situations involving triangles. • explore and describe periodic real-world phenomena, using the sine and cosine functions. • apply general graphing techniques to trigonometric functions. • express the relationship between trigonometric and circular functions.	• Students use the local pier as their frame of reference to determine the time when a boat can safely moor. They determine from a newspaper when high and low tides occur, and study depth charts or talk with the harbor master to determine the depths at high and low tide. Students find a trigonometric equation that models the depth of the water and use graphing calculators or computers to help them find the best times for mooring.

2.12 Conceptual Underpinnings of Calculus

Grades 11–12 Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • investigate and describe limiting processes by examining infinite sequences and series. • determine maximum and minimum points of a graph and interpret the results in problem situations. • investigate and describe limiting processes by examining areas under curves.	• Students explore limiting sums for sequences of fractions and repeating decimals. They begin their explorations by shading grid paper and/or folding and cutting squares of paper to estimate the limits. After making conjectures about formal procedures for finding sums, they use graphing calculators to verify or refine these procedures. 

2. PATTERNS, RELATIONS, AND FUNCTIONS
LEARNING STANDARDS FOR ADULT BASIC EDUCATION

2.13 Patterns, Relations, and Functions

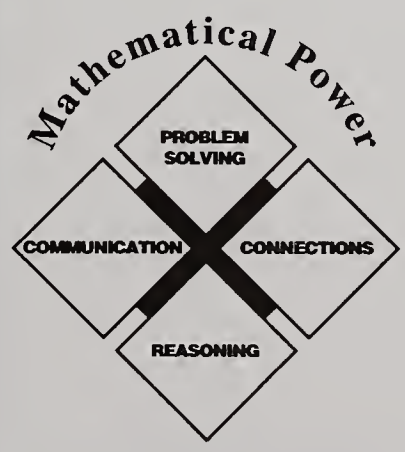
ABE Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • explore, identify, analyze, and extend patterns in mathematical and real-world situations. • articulate and represent number and data relationships, using words, tables, graphs, and rules. • discover and use patterns and functions to represent and solve problems.	• Using a spreadsheet, students explore the rules of operations with integers. They see what happens when they combine positive and negative numbers, using the four basic operations. After recording the results of their experimentation, students form groups to discuss why the procedures work as they do. They brainstorm to find real-life examples that might help them make sense of their results.

2. PATTERNS, RELATIONS, AND FUNCTIONS LEARNING STANDARDS FOR ADULT BASIC EDUCATION

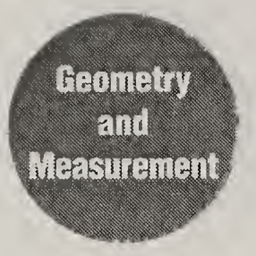
2.14 Algebra

ABE Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • represent arithmetic patterns and real-world situations, using tables, graphs, verbal rules, equations, then explore and describe the interrelationships of these representations. • recognize and use variables, expressions, and equations. • solve equations in one or two variables, using concrete, informal, and formal methods. • apply algebraic methods to solve and represent a variety of test-related, work-specific, and real-world mathematical problems.	• Students determine how far they live from class and record how long it takes each of them to get there from home. The class makes a chart that lists the distances and times, then uses the distance formula to determine each student's rate of travel. Students discuss reasons why some rates might be similar or different.

How It Looks in the Classroom



In a pre-General Education Development (GED) classroom, students explore the relationship between the weight and the volume of water, using a set of calibrated measuring cups. They record their findings and write a rule that summarizes those findings. As a follow-up, students and teachers bring to class a variety of containers, measure the dimensions (to determine the volume) of each container, and then use their rule to predict the weight of water. Finally, each container is filled with water and weighed to check their predictions.



STRAND 3:
GEOMETRY AND MEASUREMENT

Geometry and measurement help us represent in an orderly fashion what we see in our world. Whether we are cooking or cartooning, shopping or shipping, painting a canvas or a wall, designing space craft for NASA or for preschool, we continually bump up against these mathematical organizers. Lifelong learners should know and understand these interconnected and symbiotic mathematical domains.

Before students begin school, they already have developed a knowledge of and a curiosity about the physical and spatial world around them. They are exploring size, shape, and position when they buy new sneakers, or select fruit at the farmer's market, or play Simon Says. Teachers can use this natural curiosity and real-world experience to begin helping students build their own mathematical foundation in geometry and measurement. Geometry and measurement experiences that are presented in problem-solving situations and incorporated into number patterns and other mathematical ideas increase their understanding of mathematics in real-world contexts.

Tools for Understanding

In the early grades, informal and intuitive studies of geometry and measurement are appropriate for tapping students' natural curiosity and engaging them. Students usually measure objects with nonstandard units, such as shoes or hands, or strides. These informal, often gross motor, explorations of spatial sense, size, and shape make young learners' mathematical experiences personal and fun. Through them, students discover for themselves the necessity of standard units.

As students explore concepts and ideas with the aid of manipulatives such as blocks, geoboards, balance scales, and measuring tools they develop a personal and sensory understanding of mathematical ideas.

Dynamics and Dimensions

Continuing the study of geometry and measurement in the middle grades allows students to see for themselves the dynamic role that mathematics plays in the environment. As students use two- and three-dimensional models to investigate patterns and to develop spatial skills, they uncover some of the practical uses of mathematics. Ways in which mathematics is used in a wide variety of careers become believable.

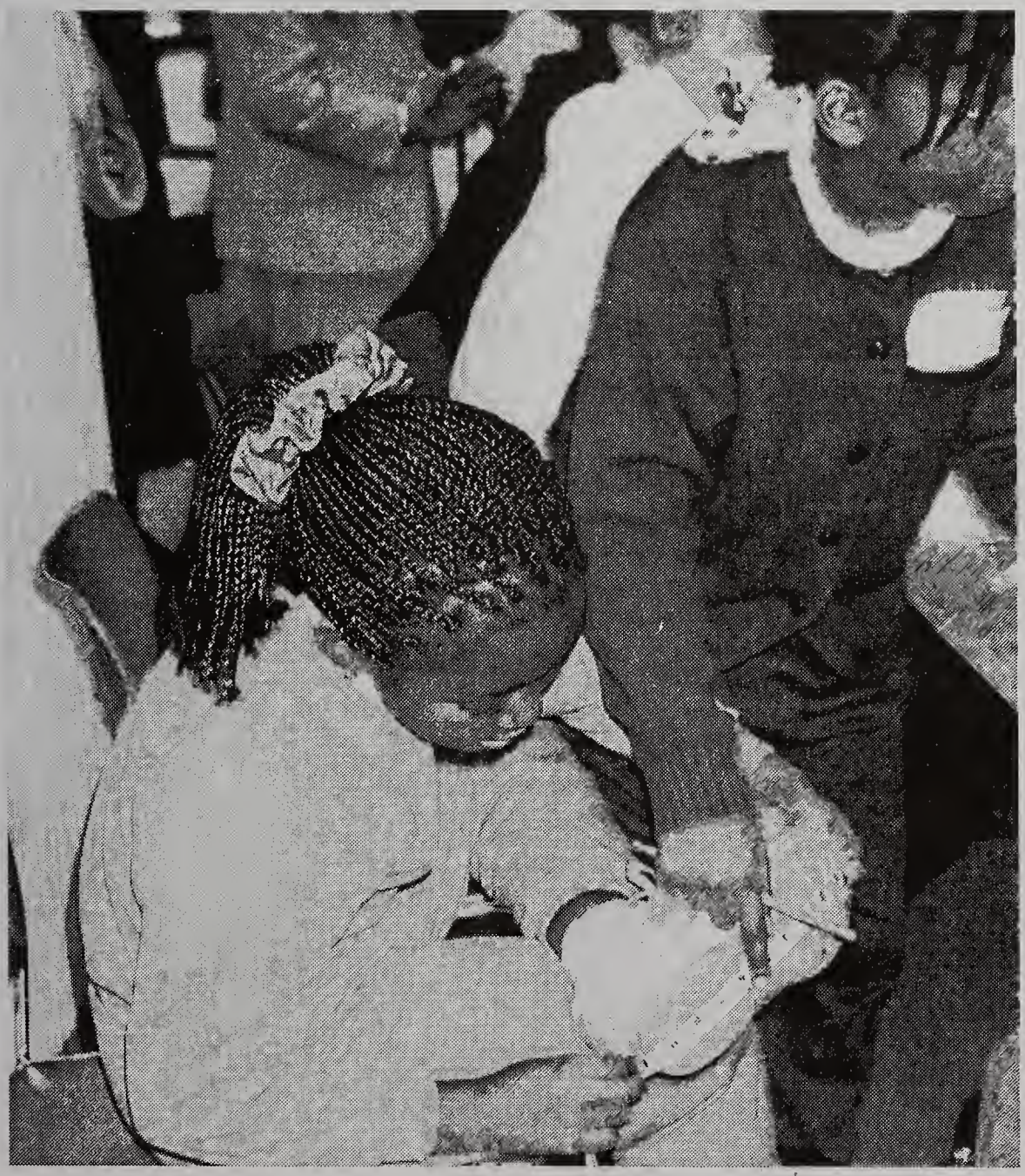
Geometry and measurement often spark a renewed interest in mathematics for those students who have been turned off for some reason, or heretofore have felt unsuccessful with mathematics learning. Investigating problems that involve geometry and measurement broadens all students' mathematical understanding and engages them as they explore mathematical ideas.

Argumentation and Visualization

As the study of geometry and measurement becomes more formal in high school, students become better equipped for mathematical argumentation. They should have opportunities to develop and defend their conclusions in verbal debates and written essays with emphasis on traditional methods of proof. Instead of emphasizing only two-column proofs, students should also be given opportunities to justify their own conclusions with less formal, but nonetheless convincing, arguments. Students' spatial reasoning and visualization skills should be enhanced. The study of geometry should make full use of all available technology.

Building Comprehension and Vocabulary

Hands-on, interactive investigations, using nonstandard and standard units, help adult basic education students develop an understanding of the many measurable attributes of physical objects. Measurement sense including length, time, temperature, capacity, weight, mass, area, volume, and angle will benefit from this approach. This realistic approach helps build an accessible measurement vocabulary and a meaningful comprehension of what it means to measure.



Hands-on investigations help enhance students' vocabulary and comprehension.

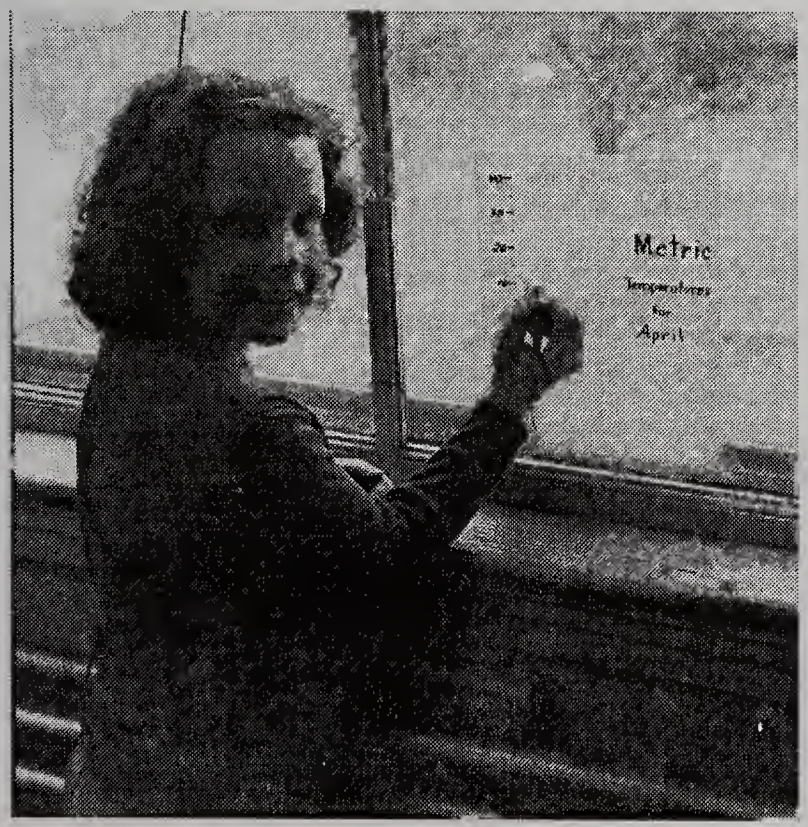
3. GEOMETRY AND MEASUREMENT LEARNING STANDARDS FOR PREK – GRADE 4

3.1 Geometry and Spatial Sense

PreK–4 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • describe, model, draw, and classify shapes. • investigate and predict the results of combining, subdividing, and changing shapes. • develop spatial sense. • use geometric ideas to develop numerical ideas. • recognize and appreciate geometry in the world.	• Place a shape inside a covered box. Pass it around. Have each student feel, then describe some of the characteristics of the shape. Record. When everyone has contributed, name the shape. Review the characteristics and see which characteristics always meet the criteria for the shape. • Students design quilt blocks by drawing contiguous geometric shapes on graph paper and shading or coloring them. As an extension, they transfer their design to cotton fabric and assemble a quilt block, which can be finished as a pot holder, or combined with other students' blocks to form a geometric wall hanging for the classroom. A field trip to the New England Quilt Museum in Lowell enhances the historic aspect of the activity. • Students work in groups with the traditional seven piece tangram, trying to see whether they can make each of the tangram shapes (triangle, square, parallelogram) using 1, 2, 3,...7 pieces. They record their findings, using drawings and charts, and give a presentation to the class that includes their reasoning to justify their findings concerning shapes that may be impossible to form. This activity can be extended to include work with fractions and ratios by assigning unit values to different shapes. Students can make their own tangram puzzles. Connections with literature and other cultures can be made by exploring the historical background of the tangram puzzle.

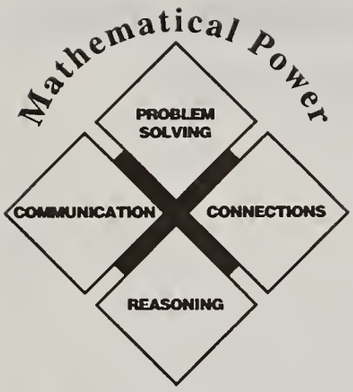
3.2 Measurement

PreK–4 Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • demonstrate the attributes of length, capacity, weight, area, volume, time, temperature, and angle. • use the process of measuring and the concepts related to units of measurement. • make and use estimates of measurement. • measure in everyday problem situations.	• Working in small groups, students make boxes from pieces of paper and explore relationships. They have available to them a myriad of materials and manipulatives, including grid paper for graphing and cutting, cubes or beans or rice or candy bits, graduated cylinders, tape, and scissors. Students try to find the box with the largest volume and the smallest volume. Is this the same as the box that will hold the most or least number of cubes, beans, rice or candy? As they work with the materials, students explore different relationships as they convince others that their box holds the most.



Knowing and being able to use metric measurement is essential in many careers.

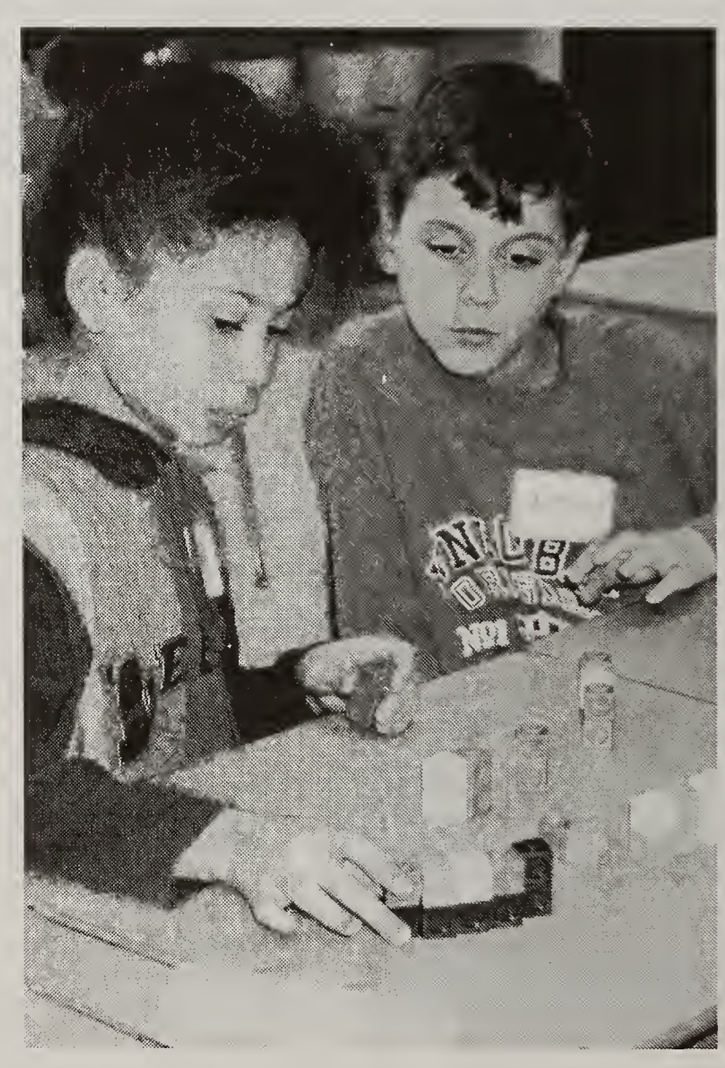
3. GEOMETRY AND MEASUREMENT LEARNING STANDARDS FOR PREK – GRADE 4



How It Looks in the Classroom

Students in a second grade class explore 3-dimensional figures made from wood by the grandfather of one of the students. First they observe and feel the differences, then verbally compare the objects in cooperative groups. Students are asked to construct shapes in any way they wish. The shapes to be constructed are cubes, cylinders, triangular prisms, rectangular prisms, and spheres. Students become creative in solving the problem of how to construct a sphere. Some students use a partially inflated balloon, others make wads of aluminum foil. Still others overlap and connect metal rings for holding loose-leaf paper.

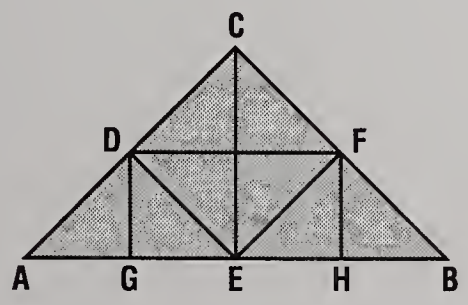
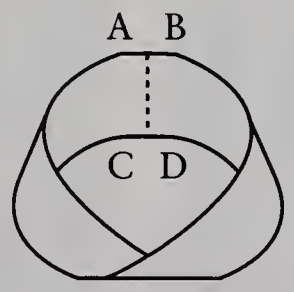
As a culminating activity, the many shapes are labeled and hung together to make mobiles, which are hung in the hallway for everyone to see.



Learning by doing engages students in the discovery of properties of 3-dimensional objects.

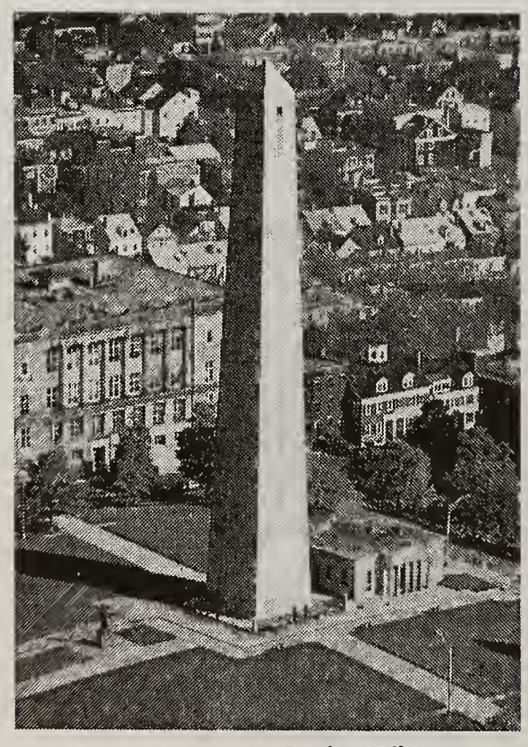
3. GEOMETRY AND MEASUREMENT LEARNING STANDARDS FOR GRADES 5 – 8

3.3 Geometry

Grades 5–8 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • identify, describe, compare, and classify geometric figures. • explore and describe the properties of points, lines, and planes. • visualize and draw geometric figures. • explore and describe transformations of geometric figures. • represent and solve problems, using geometric models. • apply geometric properties and relationships. • develop and explain the concept of π. • develop and explain the concept of the Pythagorean theorem.  	• Students build shapes with toothpicks and 1 cm clay balls. They make geopanes by hanging the shapes on thread and dipping them into a mixture of water and soap. Beforehand, students predict what they think will happen and describe what does happen. This illustration of surface tension encourages discovery of the relationship of vertices, edges, and faces of 3-dimensional figures and an understanding of Euler's theorem. • Students build a scale model of their school. They sketch elevations of the school building, estimate and measure the dimensions of the sides, determine the most cost-effective materials for construction of the model, and write a presentation describing their model. The model is displayed and the presentation is delivered to school and local communities. • Students make Möbius strips to model a continuous one-sided surface. They write about what they observe and why they believe it is that way. 

3. GEOMETRY AND MEASUREMENT LEARNING STANDARDS FOR GRADES 5 – 8

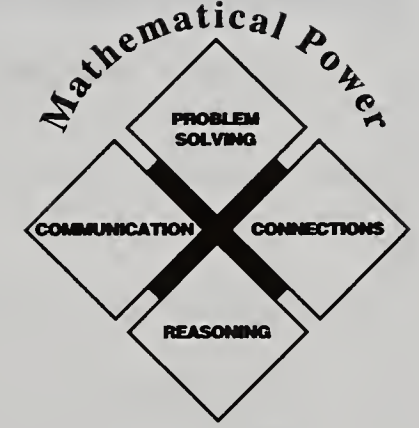
3.4 Measurement

Grades 5–8 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • select appropriate units and tools to measure to the degree of accuracy required in a particular situation. • describe the meaning of perimeter, area, volume, angle measure, capacity, density, weight, and mass. • develop and describe the concepts of rates and other derived and indirect measurements. • develop and apply formulas and procedures for determining measures to solve problems.  *Bunker Hill Monument*	• Monuments found in Massachusetts provide the basis for exploration of indirect measurement and geometric constructions. At home, students peruse collections of post cards and photographs to find those with shadows cast by tourists and monuments. At school, groups of students cull the photographs and use them to derive the heights of different monuments. Students use measurement to construct clay and sand replicas of pyramids or obelisks are constructed and measured. Formulas are used to derive volume. A connection with art is reinforced with a trip to the Daniel Chester French house museum called Chesterwood in Great Barrington. It is a tribute to one of the most notable monument sculptors of our region. • Students construct containers, explore surface area, and use popcorn to explore volume relationships. They sketch the container they believe to hold the most popcorn. They use surface area to find the most economical container to hold a specific volume.

3. GEOMETRY AND MEASUREMENT LEARNING STANDARDS FOR GRADES 5 – 8

How It Looks in the Classroom

To explore the constant relationship between the diameter of a circle and its circumference, a class of sixth graders form a straight line in a snowy field, with the teacher in the middle. As the line rotates around the teacher, a circle is formed by footprints in the snow. The class lines up around the circumference and predicts how many times their length will fit around the circle. By marking off start and finish points as the class "diameter" advances along the circumference, students discover that a little more than three diameters fill the circumference. They hypothesize whether this would be different for a larger or smaller class. Later in the art room, they test their ideas by using potato prints and units to measure the diameter and circumference of different size circles.



3. GEOMETRY AND MEASUREMENT LEARNING STANDARDS FOR GRADES 9 – 10

3.5 Geometry and Spatial Sense

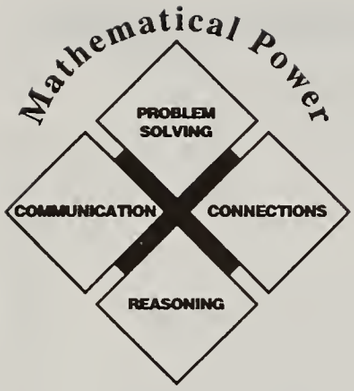
Grades 9–10 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • interpret and draw three-dimensional objects. • represent problem situations with geometric models and apply properties of figures. • classify figures in terms of congruence and similarity and apply these relationships. • deduce properties of, and relationships between, figures from given assumptions. • develop and defend conclusions. • formulate counter examples. • construct proofs for mathematical assertions, including indirect proofs and proofs by mathematical induction.	• Students examine *Ascending and Descending* by Maurits Escher. They focus on the staircase in the picture and individually in writing, describe relationships between the points, lines, and planes that portray the appearance desired by the artist. Then they make their own geometric "Escher" drawings and describe the geometry they used to make them. • Students use congruent right triangles cut from a square to prove the Pythagorean theorem both geometrically and algebraically. • Students generate and solve problems involving the shortest paths for interconnecting three or more points. They use geometric constructions (both compass/straightedge and computer programs) to locate possible points, and find distances and angle measures. The students generalize their results and form conjectures regarding the location of new points in the network that will provide the solution.

3. GEOMETRY AND MEASUREMENT LEARNING STANDARDS FOR GRADES 9 – 10

3.6 Geometry from an Algebraic Perspective

Grades 9–10 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • translate between synthetic and coordinate representations. • deduce properties of figures, using transformations and coordinates. • identify congruent and similar figures, using transformations. • develop and explain geometric interpretations and applications of slope.	• Students fold a rectangular sheet of construction paper once diagonally and cut it to make a square. They open the square, then fold it into fourths, representing four quadrants. With their paper open and the diagonal fold in the lower left corner, students use colored chalk to draw any graph they wish in the upper two quadrants. They fold over their square, along the horizontal fold, press and rub with their palm. When students open it, they describe what is represented. • Students investigate slope by developing plans for increasing handicap access to buildings, by investigating state and local codes for ramps and problems that occur in their construction.

3. GEOMETRY AND MEASUREMENT LEARNING STANDARDS FOR GRADES 9 – 10

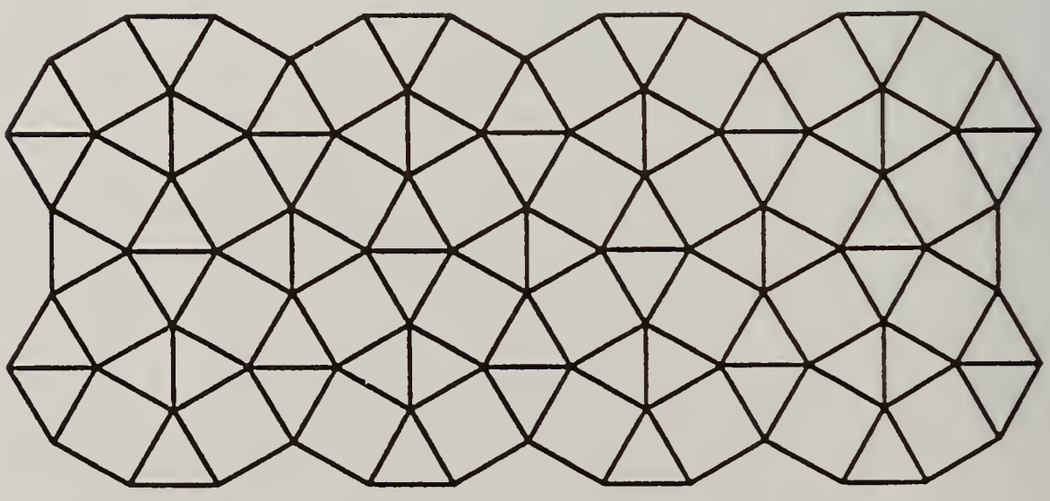


How It Looks in the Classroom

Students in Ms. Rivers ninth grade class are studying many topics of mathematics combined with other disciplines through the study of tessellations. They explore the work of the Dutch artist M.C. Escher, and historical examples of tilings, such as the Islamic tiling found in the Alhambra.

They begin by exploring polyominoes (using one, two, three, four, five, and, for those who want a challenge, six squares), looking for patterns; learning about reflections, translations and rotations; and deepening their spatial visualization skills.

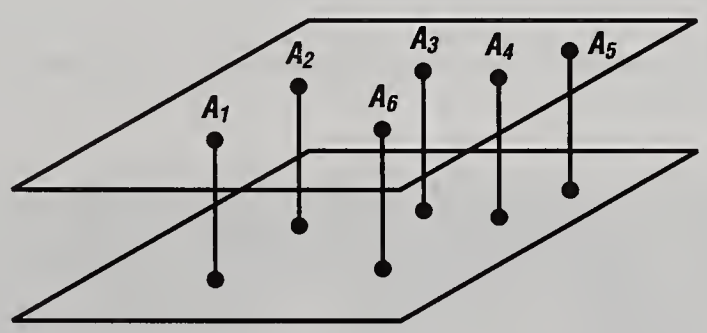
Students use many different materials and media to arrive at and present their findings. Some students write and use a Logo program, while others do their work on grid paper, and others build models and trace their results. As results are presented, new questions are raised regarding functional relationships, with students trying to find either an explicit or recursive expression for the relationships. They use graphing calculators to try to answer their questions. After the presentations, students play a computer game involving shapes made up of four squares with new understanding of the mathematical connections.



Tessellations have many real-life applications that integrate mathematics with other disciplines.

3. GEOMETRY AND MEASUREMENT LEARNING STANDARDS FOR GRADES 11 – 12

3.7 Geometry and Spatial Sense

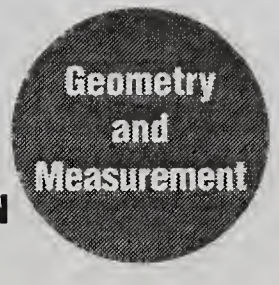
Grades 11–12 Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to:	
• deduce properties of, and relationships between, figures from given assumptions.	• Topology and networks are explored by means of a manipulative designed by the teacher. With the aid of the vocational-technical school, two pieces of Plexiglas are anchored with three randomly placed nuts and bolts. When the Plexiglas is dipped in a soap solution, a network is revealed, demonstrating a Steiner point. This manipulative can be used on an overhead or placed atop a map to illustrate networking problems such as those addressed by the airlines.
• use vectors, phase shift, maxima, minima, inflection points, and precise mathematical descriptions of symmetries to locate and describe objects in their orientation.	

3. GEOMETRY AND MEASUREMENT LEARNING STANDARDS FOR GRADES 11 – 12

3.8 Geometry from an Algebraic Perspective

Grades 11–12 Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to:	
• apply transformations and coordinates when solving problems.	• Students graph separately several functions on grid paper. Using a mirror, they reflect each function, in the line y = x, and sketch their results. Students list some values of x and fill in a table by reading the y coordinates from the graph. They try to derive the relationship between the x- and y- coordinates of the new graph. They write the new function and compare it with the original. After students have done several pairs of functions, they should see that the inverse of a function can be found by reflecting about the line y = x.

High school students use mirrors to see the inverse of a function.



3.9 Geometry and Spatial Sense

ABE Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • use geometry to describe the physical world. • apply geometric properties and relationships to concrete situations. • visualize and represent geometric figures. • identify, describe, compare, and classify geometric figures. • relate geometric ideas to number and measurement ideas, including the concepts of perimeter area, volume, angle measure, capacity, weight, and mass. • explore and describe transformations of geometric figures. • represent and solve problems, using geometric models. • use a variety of technologies to study geometry and spatial sense.	• Each student works on every aspect of designing and making a quilt square. They design a pattern, buy the fabric, cut out their pieces, and sew the square. Students explore relationships between shapes and angles, find a range of challenging levels of geometric complexity, and solve the many problems that arise as they transform their abstract design into a concrete product. Finally, the quilt squares are assembled into a lap quilt, which can be donated to a local hospital.

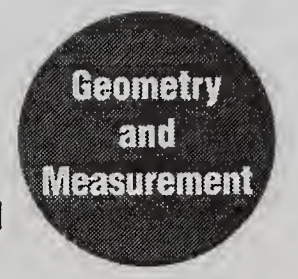
3. GEOMETRY AND MEASUREMENT
LEARNING STANDARDS FOR ADULT BASIC EDUCATION

3.10 Measurement

ABE Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • make and use exact and estimated measurements to describe and compare phenomena. • select appropriate units and tools to measure to the degree of accuracy required. • use systems of measurement.	• Students use an assortment of measurement tools, including rulers, thermometers, gas gauges, and speedometers. They learn to read and interpret both labeled and unleveled calibrations for tools used in a variety of workplace settings.

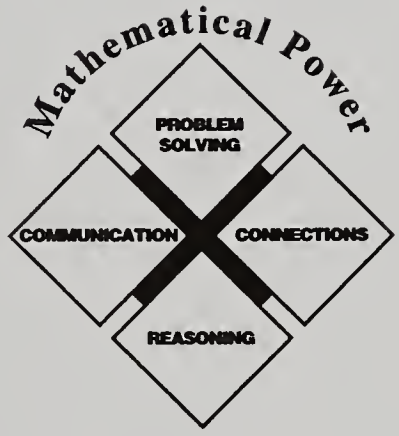
Adult learners read and interpret a measurement tool by means of using that tool in a workplace environment.

3. GEOMETRY AND MEASUREMENT LEARNING STANDARDS FOR ADULT BASIC EDUCATION



How It Looks in the Classroom

At one Massachusetts community college, a teacher collected measuring cups, pint containers, empty quart-size milk cartons, and gallon jugs to set up a liquid measurement lab in her classroom.

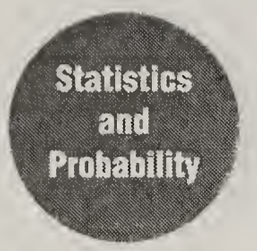


Students poured water and discovered drop-for-drop that eight ounces filled one cup. They poured cups of water into a pint carton and saw for themselves that one pint equaled two cups. It was messy, and it took more class time than asking students to memorize a table of measures.

Some students said after class that for the first time they could see what a pint was, that they understood measuring, and could picture equivalencies in their heads. They felt that this idea was a good one and hoped to do more labs. One 16-year-old boy, who has lived in foster homes most of his life, said that he had never used a measuring cup before.

When Roberta came to class the next day, she told the class what she had learned. "I never realized that when I gave my son eight ounces of formula that he was drinking a whole cup of formula."

Counting systems directly affect measurement recording.



STRAND 4:
STATISTICS AND PROBABILITY

"All students should analyze, develop and act on informed opinions about current economic, environmental, political and social issues affecting Massachusetts, the United States and the world."

Massachusetts Common Core of Learning

Statistics and probability confront us every day. The ability to understand variability and uncertainty is necessary every time one reads the results of a Gallup poll or a report on the latest medical research findings. What does it mean to test positive for HIV? What are the chances that a nuclear power plant will have an accident? Is angioplasty as effective as coronary artery bypass surgery? If we are to interpret the arguments made by those on each side of an issue and make our own informed decisions on questions like these, then we must rely on our understanding of statistical inference, misuses of statistics, and probability.

Level by Level

Sorting and classifying by attributes are likely to be the first formal introduction to statistics that a student encounters. At a very young age, students begin to describe, analyze, evaluate, and make decisions about the attributes of familiar items such as blocks, shapes, or clothing. Students in early grades should explore probability and data analysis by working with one variable and learning to make, read, and interpret simple graphs. Students learn that a sample can be representative of a population. The more opportunities students have to do hands-on activities with probability and data, the better base they have from which they develop a deeper understanding.

In the middle grades, students use scatter plots to explore the relationship between two variables, construct circle graphs to represent proportional amounts and histograms for data that are grouped into intervals, and construct box and whisker plots to represent data spreads. They also explore differences between theoretical and experimental probability. They extend their data analysis and probability repertoire to include sampling bias and randomness.

Students in high school extend their knowledge to fitting nonlinear graphs to data. They study in depth sampling methods and the role of sampling in making predictions and judging the validity of statistical claims. The topics explored should be integrated with other mathematics courses.

Role of Critical Thinking

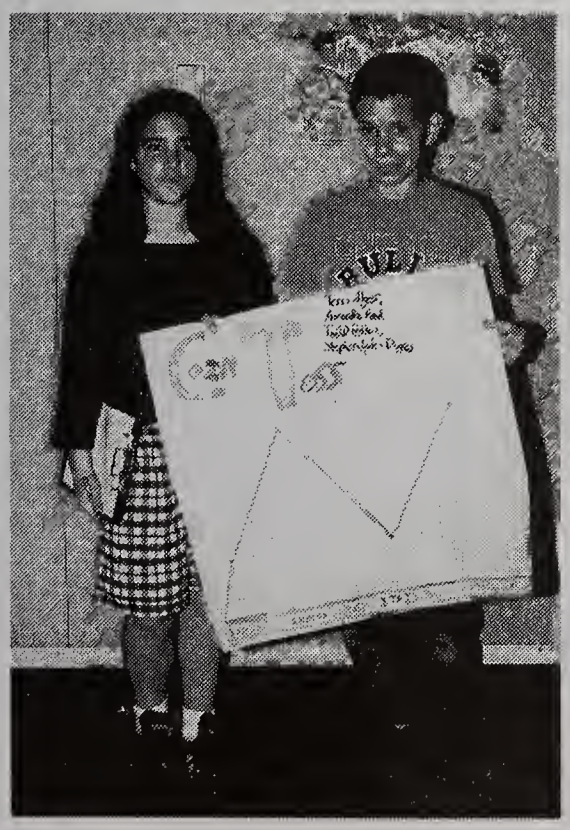
The development of critical thinking in statistics should be emphasized. Students at all levels should formulate appropriate questions; gather and explore data; organize and describe data, using graphs, charts, and tables; interpret results; and develop a critical attitude toward the use of statistics. They should investigate real-life problems that require them to employ sampling techniques. These investigations help them realize the relative applicability of statistics to solving problems.

Emphasis on Interpretation

Technology has changed dramatically the way we deal with statistical data. Statisticians spend relatively more time interpreting what their data suggests, using exploratory techniques, and relatively less time applying standard inferential techniques. What computers have done for statisticians, inexpensive statistical calculators and classroom software have done for students. More emphasis should be placed on interpretation of summary statistics—both student-generated and from daily life.

Adult Investigations

Adult learners should become keenly aware that decisions made on the basis of statistics affect them daily. Involve them in discussions about average scores required for tests, percentage of attendance at work or school, weekly price of transportation, cost of child or health care, or the number of immigrants allowed to become legal residents. Adult mathematics investigations should evolve around such statistical representations. They should understand how statistical information may be misused.



Probability and teamwork go together for these young mathematicians.

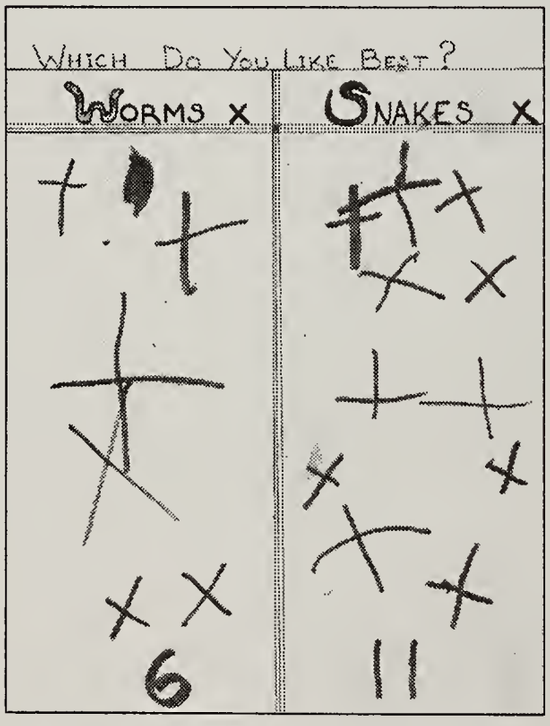
4. STATISTICS AND PROBABILITY LEARNING STANDARDS FOR PREK – GRADE 4

4.1 Statistics and Probability

PreK–4 Learning Standards

Students engage in problem solving, communicating, reasoning, and connecting to:

- collect, organize, and describe data.
- construct, read, and interpret displays of data.
- formulate and solve problems that involve collecting and analyzing data.
- explore and describe the concepts of chance.



Statistics and probability can be introduced during early childhood years.

Examples of Student Learning

- Give each pair of students 25 assorted pattern blocks. Have them sort the blocks and design a representation of the sorting. Discuss how they sorted and constructed their representations.
- Students make a floor graph that displays data gathered from classmates. The teacher marks with masking tape on the floor two outlines of bars as in a bar graph. Students each trace one foot on paper and cut it out. Next, they place their cutouts on the bar that identifies whether each is wearing sneakers or not wearing sneakers. Discuss what the graph shows. Extend the activity by gathering the data by using tally marks, and graphing the findings on inch-square graph paper.
- Students check newspapers and find the time of sunset for Monday through Friday. The data are recorded in a table and a line graph is constructed. Students interpret their graphs and predict the times of sunset for the following week. They record them on the graph in a contrasting color. They check their predictions in the newspaper and discuss their conclusions. Is there a pattern? If so, what is it?
- Groups of four children choose four different types of candy they will use in a survey. Each group makes a chart to show the possible selections. Each group surveys another class. One student explains the survey, another explains the selection chart, one counts the votes, and another tallies the votes. The groups graph their data.
- Place two apples in a bag—one green apple, one red apple. Have the students predict what will happen when choosing an object always? never? sometimes? Next, have the students predict which color apple will come out most often. Investigate. Discuss.

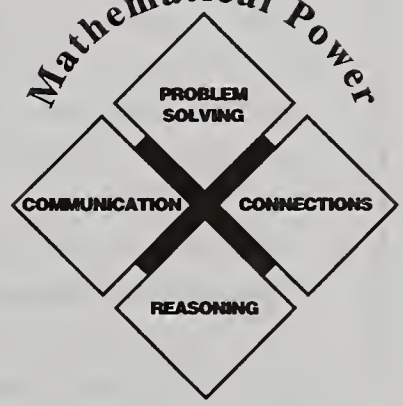
4. STATISTICS AND PROBABILITY LEARNING STANDARDS FOR PreK – GRADE 4

How it Looks in the Classroom

A class of second grade students has been working with an integrated mathematics and science program, which introduces the concepts of attributes, estimation, and probability. Each is working with a partner to play a game. Students select a number from 1 to 12. Each pair takes two foam dice and 12 frog playing pieces with a game board. Directions are given to throw the dice, add the two numbers, and move the frog whose number matches the total, one space on the board. The game continues until a frog reaches the other side of the "pond."

As play begins, Briana and Greta observe that frog 1 will never win. The teacher asks why. They respond that since players must throw two dice, the lowest number possible is 2, and therefore frog 1 will never get a chance to move forward.

The teacher keeps track of the winning numbers on the board with stick-on tabs, eventually forming a graph showing frogs 6, 7, and 8 winning more often than the others. Again, she asks the students why they think this is so. Remembering what Briana and Greta said about frog 1, Justin suggests that the only way a player can move frog 3 is by rolling a 1 with a 2. The teacher challenges the class to determine all the combinations that will give a total of 7. The students discover 1 + 6, 5 + 2, and 3 + 4. Since there were more possible combinations for 7, they concluded it would probably be rolled more often. When the students next play the game, most choose 6, 7, or 8 as the winning frog!



Students create a human graph by physically representing the data.

4.2 Statistics

Grades 5–8 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • collect, organize, and describe data systematically. • construct, read, and interpret tables, charts, and graphs. • make inferences and convincing arguments that are based on data analysis. • evaluate arguments that are based on data analysis. • develop and explain why statistical methods are powerful aids for decision making.	• After reading an article, students interpret the data and evaluate the conclusions expressed by the reporter. Students write letters to the editor, questioning or approving the reporter's conclusions. Students find other articles that use statistics and their interpretations and discuss them with someone at home. • Students conduct research to learn more about the Consumer Price Index (CPI)—how it is expressed, what it tells us, and how it is determined. Through whole class discussion, they analyze its appropriateness. Probing questions are asked in an attempt to prompt students to realize that the CPI does not take into account the decisions that people make based upon price changes. Working in groups, students create their own measurement of the cost of living for a middle grades student. Will they accommodate price changes? How? What costs will they consider? Will they estimate or be exact? Why? Groups will prepare and deliver oral presentations of their ideas.

4. STATISTICS AND PROBABILITY LEARNING STANDARDS FOR GRADES 5 – 8

4.3 Probability

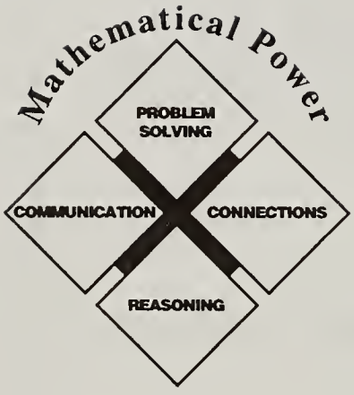
Grades 5–8 Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • model situations by devising and carrying out experiments or simulations to determine probabilities. • construct a sample space to determine probabilities. • describe the power of using a probability model by comparing experimental results with mathematical expectations. • make predictions that are based on experimental or theoretical probabilities and determine their reasonableness. • develop and explain an appreciation for the pervasive use of probability in the real world.	• At the beginning of every mathematics class, each student picks a four-digit number and records it on a chart. At the end of class, the teacher selects a four-digit number by randomly drawing numbered tiles from a bag containing ten tiles numbered from 0–9. After each draw, the tile is returned to the bag before the next draw. A winning number has any three digits. Keep track of the number of times there are winners. Periodically discuss informally with the class what they are discovering about chance and why they believe it is so. How would they change the classroom lottery to make more winners? Fewer winners? What happens to the odds if each tile is not returned to the bag?

How It Looks in the Classroom

Eighth graders at an urban school in the Commonwealth devised their own survey questionnaire. Questions ranged from topics of general interest (height, foot size, number of siblings, favorite shoe brand, and type of music) to those relevant to their urban experience (curfew, suspension from school, rules at home). They filled out the questionnaire during one class period, while taking turns at height and foot measuring stations. To display categorical data, a hypothetical frequency table was made on the board.

Favorite Music	Number of students
Reggae	9
Rock	2
Rap	8
Other	6

One student suggested displaying the results in a bar graph, and the class discussed ways to construct, label, and title such a graph. Several times during this discussion, Shandra insisted a pie chart would be better, so the discussion turned to the subject of pie charts.



Teacher: How do we construct a pie chart?

Student: Draw a circle. [The teacher draws a circle on the board.]

Teacher: How do we divide up the pie?

Student: Reggae gets the largest piece. [The teacher shades in a piece roughly equal to three-quarters of the pie.]

Teacher: Is that okay?

Student: No!

Teacher: Then how do I decide what portion of the circle to shade for reggae?

Student: Guess.

Teacher: I did guess.

Student: You can guess better than that.

Teacher: Maybe I can't. Why don't each of you draw the section as you guess it should be, and we will check them after we figure this out more precisely. [Students draw what they think will be reggae's portion of the circle graph.] Now that we have guessed, tell me a way we could divide the pie that uses mathematics.

Student: Well, a circle is 360.

Teacher: A circle is 360 what?

Student: 360 degrees.

Teacher: The entire way around the circle is 360 degrees. That's a good starting point.

Student: Where is that protractor you gave us? [He refers to a giant cardboard circle marked in degrees, used during discussion of angles in geometry. We locate this protractor and place it in front of the room.]

Teacher: Now what?

Student: There are 25 students.

Teacher: What do we do with 25?

Student: Divide; divide 360 by 25. [Teacher writes the problem on the board.]

Student: But 25 won't go into 360 evenly.

Teacher: That's okay, we know how to use decimals. [There are lots of puzzled looks.] What if we had $360 and 25 students; if we divide 360 by 25, what do we have?

Student: How much money each student gets.

Teacher: Right. Now if we have 360 degrees and 25 students, what do we get? [Some make number guesses.] You don't have to tell me a numerical value. Just tell me what the number will represent.

Student: You'll get the size of a piece of pie for each student.

Teacher: Good. The size of the piece, or angle for each student. Now, how would you show that 9 students prefer reggae.

Student: Use 9 pieces.

Teacher: And rap?

Student: Use 8 pieces.

The class ends with the promise to continue the discussion of pie charts and to organize the work for analyzing their survey data. Students later create personal versions of the circular protractor with a ribbon through the middle that helps them mark out angles when constructing pie charts for their survey data.

4. STATISTICS AND PROBABILITY LEARNING STANDARDS FOR GRADES 9 – 10

4.4 Statistics

Grades 9–10 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • construct, draw inferences, and reason with charts, tables, and graphs that summarize data from real-world situations. • use sampling to recognize and describe its role in statistical claims. • design a statistical experiment to study a problem, conduct the experiment, and interpret and communicate the outcomes.	• Students bring to class examples of statistics in the media. They analyze and critique the use of the statistics, then write alternative uses of the statistics, contrasting the impression given by the advertisement with what the text actually says. For example, a recent advertisement included the statement that more than 98% of one maker's trucks sold in the last 10 years are still on the road. Students should understand that a quick reading might give the impression that 98% of the trucks sold 10 years ago are still on the road, or that all of that maker's trucks last at least 10 years. With careful reading students determine that all of the trucks still on the road might have been sold in the last two or three years. They understand that the figure quoted may be accurate or inaccurate, but that even if it is reasonably accurate it is potentially misleading as used. • Students gerrymander the state to reconfigure political party distributions so that one party gets an overwhelming majority of seats. They then work out a fair redistricting. Extend the activity by asking students to research the Massachusetts basis for the term *gerrymander.*

4. STATISTICS AND PROBABILITY LEARNING STANDARDS FOR GRADES 9 – 10

4.5 Probability

Grades 9–10 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • use simulations to estimate probabilities. • determine the likelihood of outcomes, using theoretical probabilities.	• Students drop a penny in a grid of 50 mm squares. They win if it does not cross a line, and lose if it does. They compute experimental probability and determine theoretical probability. Study can be expanded to other types of grids. • Students use probability to determine fairness of a group birthday game. Two guests enter a room of people and ask everyone to write their birthdays on a piece of paper. If two people have the same birthday, the first guest wins. If no two people have the same birthday, the other guest is the winner. How many people should be in the room for this to be a fair game? Students first guess the number of people that should be in the room, then try it with their class. Finally, they explore it further with the aid of a computer program.

4.6 Statistics

Grades 11–12 Learning Standards	Examples of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • use curve fitting to predict from data. • apply measures of central tendency, variability, and correlation. • design a statistical experiment to study a problem, conduct the experiment, and interpret and communicate the outcomes. • analyze the effects of data transformations on measures of central tendency and variability. • transform data to aid in data interpretation and prediction. • test hypotheses by using appropriate statistics.	• Students each have two register tapes from grocery stores. They plot the number of items and total cost on the tapes as ordered pairs on a rectangular coordinate system. They draw a straight line through the middle of the points on the graph by using a clear plastic ruler, a piece of uncooked spaghetti, or some other device that allows them to see the points on both sides of the line. They adjust the device until the line goes through the center of the scatter of points and there are approximately the same number of points on each side of the line. Students then determine an equation for the line they drew by finding two points on the line whose coordinates can be read fairly accurately. In small groups, they compare lines and discuss reasons for the differences. To extend the activity, students use a statistical technique, such as the least squares regression or the median fit line, to find the line that best fits the data, rather than fitting a line to the data by eye. Students use graphing calculators or computer software to fit the line to the data. • Students can try fitting nonlinear functions such as quadratic, exponential, or logarithmic curves to sets of data, which connects statistics with algebra.

4. STATISTICS AND PROBABILITY LEARNING STANDARDS FOR GRADES 11 – 12

4.7 Probability

Grades 11–12 Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • use experimental or theoretical probability, as appropriate, to represent and solve problems involving uncertainty. • apply the concept of a random variable. • create and interpret discrete probability distributions. • describe, in general terms, the normal curve and use its properties to answer questions about sets of data that are assumed to be normally distributed.	• With the aid of a computer, students explore the rush at their local post office during winter holiday season, using experimental and theoretical probability generated by random numbers in simulated situations. Modeling is used when students interpret random numbers and simulate queues.

Technology enhances the depth and the complexity of the mathematics students explore and learn.

4. STATISTICS AND PROBABILITY LEARNING STANDARDS FOR ADULT BASIC EDUCATION

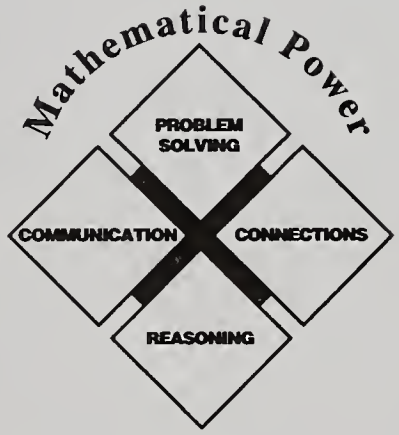
4.8 Statistics and Probability

ABE Learning Standards	Example of Student Learning
Students engage in problem solving, communicating, reasoning, and connecting to: • collect, organize and describe data. • construct, read and interpret tables, charts and graphs. • make inferences and convincing arguments that are based on data analysis. • evaluate arguments that are based on data analysis. • develop and explain an appreciation for statistical methods as a powerful means for decision making.	• Students speculate on the percentages of men and women, ethnic groups, or age groups that appear in magazine advertisements. After collecting data and presenting their conclusions, students write about marketing/advertising bias.

4. STATISTICS AND PROBABILITY LEARNING STANDARDS FOR ADULT BASIC EDUCATION

How It Looks in the Classroom

At a Massachusetts adult learning center, students conducted a survey. It asked respondents to choose from a list those three issues that concerned them the most. Issues were grouped into the categories of home, school, and community. The survey was distributed to approximately thirty-five teachers at a staff development workshop. Completed surveys were collected, tallied, and graphed by students.



One month later, at a community mathematics forum, students gave attending parents of schoolchildren the same questionnaire. These results were graphed alongside the teachers' responses. Later, teachers and parents were given the opportunity to discuss and analyze the findings.

The Next Steps

A Plan for Implementation

Teachers and administrators will need to work together to establish a plan for designing and implementing a new curriculum—including a plan for involving and communicating with families. It will be important to include a time line and a plan for ongoing communication and coordination PreK–12. In addition, the curriculum should include a plan for program assessment and student assessment.

As we look toward the future, we recognize the need to monitor the progress of this effort, which many districts may find will require more attention, time, and funding than has been devoted to such activities in the past. There are many teachers, schools, and districts in Massachusetts who are already working diligently to bring about the changes called for in this mathematics framework. They know that the effort is arduous and that as soon as one task is accomplished, another one looms ahead. What they also can attest to is that great accomplishments began with taking small steps and building upon them, step-by-step.

Whether a teacher considers one of the guiding principles and tries to develop techniques for open-ended questioning or an administrator forms a curriculum PreK–12 and adult basic education team to research integrated mathematics programs, both have been inspired by the mathematics framework to try something new, perhaps uncharted or maybe only slightly different from what they have been doing, in the way they teach mathematics. That is what this mathematics framework is about—inspiring educators to look again at how they teach mathematics and how students learn mathematics.

Endnotes:

[1] Massachusetts Software Council. *The Switched-on Classroom,* Boston, MA: Massachusetts Software Council, 1994.

2 National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: NCTM, 1989.

[3]Adapted from Mathematics for the 21st Century of the NCTM Algebra Project.

[4]Larson, Robert E. et al. *Calculus.* Lexington, MA: D.C. Heath and Company, 1994.

Suggested Readings

Achele, Douglas B., ed. *Professional Development of Teachers of Mathematics.* Reston, VA: National Council of Teachers of Mathematics, 1994.

Association of State Supervisors of Mathematics and National Council of Supervisors of Mathematics. *Guide to Selecting Instructional Materials for Mathematics Education.* Mountain View, CA: Creative Publications, 1993.

Burrill, Gail. *Data Analysis and Statistics Across the Curriculum.* Reston, VA: National Council of Teachers of Mathematics, 1992.

California Department of Education. "Instructional Materials Criteria." *Science Framework for California Public Schools.* Sacramento, CA: California Department of Education, 1990.

— "Instructional Materials Criteria." *Mathematics Framework for California Public Schools.* Sacramento, CA: California Department of Education, 1992.

A Call for Change: Recommendations for the Mathematical Preparation of Teachers of Mathematics. Washington, DC: Mathematical Association of America, 1990.

Charles, Randall I and Edward A. Silver, eds. *The Teaching and Assessing of Mathematical Problem Solving.* Reston, VA: National Council of Teachers of Mathematics, 1988.

Cooney, Thomas J. and Christian R. Hersch, eds. *Teaching and Learning Mathematics in the 1990s.* Reston, VA: National Council of Teachers of Mathematics, 1990.

Discrete Mathematics Across the Curriculum. Reston, VA: National Council of Teachers of Mathematics, 1991.

Everybody Counts: A Report to the Nation on the Future of Mathematics Education. Washington, DC: National Academy Press, 1989.

Garland, Trudi Hammel. *Fascinating Fibonaccis: Mystery and Magic in Numbers.* Palo Alto, CA: Dale Seymour Publications, 1987.

Grouws, Douglas A., ed. *Handbook of Research on Mathematics Teaching and Learning.* Reston, VA: National Council of Teachers of Mathematics and Macmillan Publishing Company, 1992.

Hiebert, James and Marilyn Behr. *Number Concepts and Operations in the Middle Grades.* Reston, VA: National Council of Teachers of Mathematics, 1988.

Hirsch, Christian R., ed. *The Secondary School Mathematics Curriculum.* Reston, VA: National Council of Teachers of Mathematics, 1985.

Issues of Curriculum Reform in Science, Mathematics and Higher Order Thinking Across the Disciplines. Washington, DC: U.S. Department of Education, 1994.

Krulik, Stephen, ed. *Problem Solving in School Mathematics.* Reston, VA: National Council of Teachers of Mathematics, 1980.

Kuhs, Therese, ed. *Mathematics Assessment: Alternative Approaches.* Reston, VA: National Council of Teachers of Mathematics, 1992.

Learning a Living: A Blueprint for High Performance, A SCANS Report for America 2000. Washington, DC: U.S. Department of Labor. April 1992.

Lennon, Mary Beth and Barbara Walthall, eds. *Sourcebook for Science, Mathematics, and Technology Education.* Washington, DC: American Association for the Advancement of Science, 1992.

Marzano, Robert J. *A Different Kind of Classroom: Teaching with Dimensions of Learning.* Alexandria, VA: Association for Supervision and Curriculum Development, 1992.

Marzano, Robert J., Debra Pickering, and Jay McTighe, eds. *Assessing Student Outcomes: Performance Assessment Using the Dimensions of Learning Model.* Alexandria, VA: Association for Supervision and Curriculum Development, 1994.

Massachusetts ABE Math Team. *Massachusetts Adult Basic Education Math Standards.* Holyoke, MA: Holyoke Community College, 1994.

Massachusetts Department of Education. *Massachusetts Common Core of Learning.* Malden, MA: Department of Education, 1994.

Massachusetts Software Council. *The Switched-on Classroom.* Boston: Massachusetts Software Council, 1994.

Mathematical Sciences Education Board. Counting on You: Actions Supporting Mathematics Teaching Standards. Washington, DC: National Academy Press, 1991.

— *For Good Measure: Principles and Goals for Mathematics Assessment: Report of the National Summit on Mathematics Assessment.* Washington, DC: National Academy of Sciences.

—. *Measuring Up.* Washington, DC: National Academy Press, 1993.

—. *Reshaping School Mathematics: A Philosophy and Framework for Curriculum.* Washington, DC: National Academy Press, 1990.

National Association for the Education of Young Children. *Developmentally Appropriate Practice in Early Childhood Programs Serving Children from Birth Through Age 8.* Washington, DC: National Association for the Education of Young Children, 1987.

National Commission on Excellence in Education. *A Nation at Risk: The Imperative for Educational Reform.* Washington, DC: U.S. Government Printing Office, 1983.

National Council of Teachers of Mathematics. *Algebra for the Twenty-First Century, Proceedings of the August 1992 Conference.* Reston, VA: National Council of Teachers of Mathematics, 1992.

__. *Assessment Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics, 1995.

__. *Addenda Series, K-12.* Reston, VA: National Council of Teachers of Mathematics, 1992.

__. *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics, 1989.

__. *Professional Standards for Teaching Mathematics.* Reston, VA: National Council of Teachers of Mathematics, 1991.

National Resource Council. *Moving Beyond the Myths: Revitalizing Undergraduate Mathematics.* Washington, DC: National Resource Council, 1991.

National Science Resources Center. *Criteria for Evaluating Elementary Science Curriculum Materials.* Washington, DC: National Academy of Sciences.

Ralph, J. and M.C. Dwyer. *Making the Case: Evidence of Program Effectiveness in Schools and Classrooms, Criteria and Guidelines for the U.S. Department of Education's Program Effectiveness Panel.* Washington, DC: U.S. Department of Education, Office of Educational Research and Improvement, 1988.

Resnick, L.B. *Education and Learning to Think.* Washington, DC: National Academy Press, 1987.

Rowan, Thomas A. and Lorna J. Morrow, eds. *Implementing the K-8 Curriculum and Evaluation Standards: Readings from the Arithmetic Teacher.* Reston, VA: National Council of Teachers of Mathematics, 1992.

Schulte, Albert P. and James R. Smart, eds. *Teaching Statistics and Probability.* Reston, VA: National Council of Teachers of Mathematics, 1981.

Steen, Lynn Arthur, ed. *On the Shoulders of Giants: New Approaches to Numeracy.* Washington, DC: National Academy Press, 1990.

Stenmark, J. K. *Assessment Alternatives in Mathematics.* Berkeley, CA: Lawrence Hall of Science, 1989.

Stenmark, J. K., ed. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions.* Reston, VA: National Council of Teachers of Mathematics, 1991.

Willoughby, Stephen S. *Mathematics Education for a Changing World.* Alexandria, VA: Association for Supervision and Curriculum Development, 1990.

Internet Resources

Massachusetts Department of Education, http://info.doe.mass.edu

This Web site is a source for updates on the Massachusetts Curriculum Frameworks, PALMS (Partnerships Advancing the Learning of Mathematics, Science and Technology), Time and Learning, Assessment, and other state education reform issues. It provides links to national and other state education agencies.

Massachusetts Corporation for Educational Telecommunications (MCET)/Mass Ed Online (MEOL), http://meol.mass.edu/subjects/mat.html

This Mathematics Web site provides linkages to:

Math Resources on TERC's Web Site
EINet's Galaxy
Geometry Forum
Mathematics Archives Teaching Materials Page
Shell Centre for Mathematical Education
Art Galleries for Mathematicians
Bibliographies for Mathematicians
More Mathematics Resources
Worldwide Webs in Mathematics

More Internet Resources for Mathematics:

The Web site of PALMS (Partnerships Advancing the Learning of Mathematics and Science), a statewide initiative of the Massachusetts Department of Education and the National Science Foundation.

http://info.doe.mass.edu/palms/

PBS Mathline is a PBS mathematics service that uses telecommunications to complement its video services.

http://www.pbs.org/mathline/mathline.html

MEOL Massachusetts Mathematics Curriculum Framework Discussion Group

news:meol.doe.mathframe

Eisenhower National Clearinghouse (ENC) for Mathematics and Science Education

http://www.enc.org

On the ENC online home page, if you scroll down to Educational Resources and click on "math" you will find links to numerous sites.

National Science Foundation, http://stis.nsf.gov/

United States Department of Education (USDOE), http://inet.edu.gov/

National Council of Teachers of Mathematics (NCTM),

http://www.pbs.org/mathline/nctmhome.html

National Science Teachers Association (NSTA), http://www.nsta.org/

National Aeronautics Science Administration (NASA),
http://www.gsfc.nasa.gov/NASA_homepage.html

Mathematical Association of America (MAA), http://www.maa.org/

International Technology Education Association,
http://www.tmn.com/Organizations//ris/TTRA.html

Center for the Enhancement of Science and Mathematics Education (CESAME),
http://www.mcet.edu/programs/s96/6067.html

Appendix A

MASSACHUSETTS CURRICULUM FRAMEWORK STRANDS
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS AND ADULT BASIC EDUCATION

K – 12 NCTM Standards
(1) Mathematics as Problem Solving (2) Mathematics as Communication (3) Mathematics as Reasoning (4) Mathematics as Connections

Strands	K - 4 NCTM Standards	5 - 8 NCTM Standards	9 - 12 NCTM Standards
Number Sense	(6) Number Sense and Numeration (7) Concepts of Whole Number Operations (12) Fractions and Decimals (5) Estimation (8) Whole Number Computation	(5) Number and Number Relationships (6) Number Systems and Number Theory (7) Computation and Estimation	(12) Discrete Mathematics (14) Mathematical Structure
	ABE Standards: (5) Estimation; (6) Numbers, Operations, and Computation		
Patterns, Relations, and Functions	(13) Patterns and Relationships	(8) Patterns and Functions (9) Algebra	(5) Algebra (6) Functions (9) Trigonometry (13) Conceptual Underpinnings of Calculus
	ABE Standards: (7) Patterns, Relationships, and Functions; (8) Algebra		
Geometry and Measurement	(9) Geometry and Spatial Sense (10) Measurement	(12) Geometry (13) Measurement	(7) Geometry from a Synthetic Perspective (8) Geometry from an Algebraic Perspective
	ABE Standards: (10) Geometry and Spatial Sense; (11) Measurement		
Statistics and Probability	(11) Statistics and Probability	(10) Statistics (11) Probability	(10) Statistics (11) Probability
	ABE Standards: (9) Statistics and Probability		

Note: The numbers in parenthesis indicate the standard number used in the NCTM *Curriculum and Evaluation Standards for School Mathematics* and *ABE Math Standards.*

Appendix B

Partnerships Advancing the Learning of Mathematics and Science (PALMS) Criteria for Evaluating Instructional Materials and Programs in Mathematics and Science and Technology

Why was this developed?

The following criteria — Evaluating Instructional Materials and Programs in Mathematics, Science and Technology — are recommended for use by Massachusetts educators.

These criteria are designed to help districts, schools and teachers to first, reassess the strengths and weaknesses of the programs and materials they have in place, and second, to assess the strengths and weaknesses of new programs and materials being considered for implementation.

The Massachusetts Department of Education does not choose to mandate specific programs, but rather to provide tools that will help professionals to select programs that best match the specific needs of their students.

Who is it for?

While the criteria are primarily for district leaders, they also provide a useful guide for teachers as they reshape specific curriculum activities to align with the Curriculum Frameworks for Mathematics and for Science and Technology. These Frameworks, which are based upon the goals set forth in the Massachusetts Common Core of Learning, present a vision for the reform of mathematics, science and technology in the Commonwealth schools. The Frameworks also provide Guiding Principles, Habits of Mind and Content Standards, which are organized according to Core Concepts, Strands, Learning Standards, and Examples of Student Learning.

How are the criteria to be used?

The Curriculum Frameworks in Mathematics and in Science and Technology should be used in conjunction with the criteria. It is unlikely that a program will satisfy all components of the criteria. Reviewing published programs critically and becoming familiar with their particular strengths and weaknesses will help teachers and districts to make informed decisions about program selection, program modification and the use of supplementary materials.

These criteria are organized into six categories. These categories should be considered as overlapping rather than distinct. In evaluating programs and materials, it is recommended that the evaluating committee give special consideration to how all the components of the materials and programs work together to ensure that students' experiences in mathematics, science and technology are of the highest possible quality.

Appendix B

All Categories reflect the Vision, Guiding Principles, and Habits of Mind described in the Content Chapters of the Mathematics and/or the Science and Technology Curriculum Frameworks.

1. **Mathematics, Science and Technology Content**
   - Reflects the Learning Standards in the Content Chapters of the Mathematics and/or Science and Technology Curriculum Frameworks.
   - Is scientifically and mathematically correct and current.
   - Incorporates real-world science technology and/or mathematics.
   - Provides opportunities to show how a scientist, mathematician or technologist thinks.
   - Reflects the diversity of our society through activities, use of language, and illustrations.

2. **Organization and Structure**
   - Provides cohesive units, multi-day in length, that build conceptual understanding.
   - Provides for in-depth, inquiry-based investigations of major scientific and mathematical concepts.
   - Emphasizes connections among science domains technology and within mathematics.
   - Emphasizes interdisciplinary connections.
   - Incorporates appropriate instructional technology.
   - Incorporates materials that are appropriate and engaging for students of the community.
   - Includes a master source of materials and resources.
   - Includes safety precautions where needed, and clear instructions on using tools, equipment and materials.

3. Student Experiences
   - Emphasize students doing science technology or mathematics.
   - Involve students in active, inquiry-based, open-ended learning, and problem solving.
   - Involve use of manipulatives to explore, model and analyze.
   - Involve use of instructional technology to visualize complex phenomena or concepts, acquire and analyze information, and communicate solutions.
   - Provide multiple routes for students to explore concepts and communicate ideas and solutions.
   - Are developmentally appropriate and provide for diverse cultural backgrounds, abilities and learning styles.
   - Encourage collaboration and reflection.
   - Have relevance to the students' day-to-day experiences.
   - Use a variety of resources (e.g., trade books, measuring tools, information technology, manipulatives, primary sources and electronic networks).

4. Teacher Support Materials

- Provide background about the content.
- Offer ideas for how parents and community could be involved and kept informed about the program.
- Give suggestions for creating a variety of learning environments, such as cooperative learning, independent research, grouping strategies, student as teacher, learning centers, and field trips.
- Reference resource materials such as appropriate videos, file clips, reference books, software, video laser disk, long-distance learning, CD ROM, and electronic bulletin boards.
- Suggest how to adapt materials for different developmental levels of students.
- Incorporate strategies for engaging all students such as open-ended questions to stimulate student thinking, journals, manipulatives, explorations, and visual, auditory, and kinesthetic approaches.
- Include suggestions for teacher use of a variety of assessment approaches such as portfolios, journals, projects, tests, and performance assessments.

5. Student Assessment Materials

- Are free of racial, cultural, ethnic, linguistic, gender, and physical bias.
- Are oriented toward problem solving and real-world applications.
- Are embedded in the instructional program, occurring throughout the unit, not just at the end.
- Incorporate multiple forms of assessment such as: student demonstrations; oral and written work; student self-assessment; technology; teacher observations; individual and group assessments, and journals.
- Focus on the process of learning such as: predicting; modeling; making inferences; and reasoning (not just the product).

6. Program Development and Implementation

- Was designed using a research base.
- Has evidence of effectiveness, such as field test data regarding impact on student learning, behavior, and attitudes, including underrepresented student populations.
- Is flexible and adaptable to local curriculum and/or school.
- Offers training, sustained technical assistance, and long-term follow-up for teachers.

Appendix B

PALMS Quality Mathematics and Science Task Force Members

Peg Bondorew	Massachusetts Department of Education
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Appendix C

The Massachusetts Common Core of Learning

(Adopted by the Massachusetts Board of Education, July 1994)

THINKING AND COMMUNICATING

All students should:

Read, Write, and Communicate Effectively

- Read and listen critically for information, understanding, and enjoyment.
- Write and speak clearly, factually, persuasively, and creatively in standard English.
- Distinguish fact from opinion, identify stereotyping, and recognize bias.
- Read, write, and converse in at least one language in addition to English.

Use Mathematics, the Arts, Computers, and other Technologies Effectively

- Apply mathematical skills to interpret information and solve problems.
- Use the arts to explore and express ideas, feelings, and beliefs.
- Use computers and appropriate technological methods to acquire, organize, and communicate information and to solve problems.

Define, Analyze, and Solve Complex problems

- Make careful observations and ask pertinent questions.
- Seek, select, organize, and present information from a variety of sources.
- Analyze, interpret, and evaluate information.
- Make reasoned inferences and construct logical arguments.
- Develop, test, and evaluate possible solutions.
- Develop and present conclusions through speaking, writing, artistic, and other means of expression.

GAINING AND APPLYING KNOWLEDGE

Common Core of Learning

All students should:

ACQUIRE, INTEGRATE, AND APPLY ESSENTIAL KNOWLEDGE

Literature and Language

- Read a rich variety of literary works including fiction, poetry, drama, and nonfiction from different time periods and cultures, relating them to human aspirations and life experiences.
- Analyze implications of literary works, and communicate them through speaking, writing, artistic, and other means of expression.
- Know and understand the development and structure of English and other languages and how learning another language fosters appreciation of people and cultures.

Mathematics, Science & Technology

- Know and understand the major mathematical concepts such as measurement, estimation, quantity, probability and statistics; and explore the relationship of mathematics to other areas of knowledge.
- Recognize and use patterns, construct mathematical models, represent and reason about quantities and shapes, draw accurate conclusions from data, and solve, justify, and communicate solutions to problems.
- Apply the fundamental principles of the life sciences, physical sciences, earth/space sciences, and the science of technology to analyze problems and relate them to human concerns and life experiences.
- Investigate and demonstrate methods of scientific inquiry and experimentation.

Social Studies, History, and Geography

- Know and make connections among important historical events, themes, and issues; recognize the role the past has played in shaping the present; and understand the process by which individuals and groups develop and work within political, social, economic, cultural and geographic contexts.
- Synthesize and communicate information about important events and fundamental concepts in Massachusetts, United States, and world history, including historical documents such as the Declaration of Independence, Constitution, Bill of Rights, Federalist Papers, and the Gettysburg Address.
- Know important information regarding the physical environment and understand concepts such as location and place, critical features of a region, demographic trends and patterns, and the relationship between people an the environment.

Visual and Performing Arts

- Know and understand the nature of the creative process, the characteristics of visual art, music, dance, and theatre, and their importance in shaping and reflecting historical and cultural heritage.
- Analyze and make informed judgments regarding the arts.
- Develop skills and participate in the arts for personal growth and enjoyment.

Health

- Know basic concepts of human development, mental health, sexuality, parenting, physical education and fitness, nutrition, and disease prevention, and understand the implications of health habits for self and society.
- Make informed and responsible judgments regarding personal health, including avoidance of violence, tobacco, alcohol, drugs, teen pregnancy, and sexually transmitted diseases.
- Develop skills and participate in physical activities for personal growth, fitness, and enjoyment.

WORKING AND CONTRIBUTING

All students should:

Study and Work Effectively

- Set goals and achieve them by organizing time, work space, and resources effectively.
- Monitor progress and learn from both successes and mistakes.
- Manage money, balance competing priorities and interests, and allocate time among study, work, and recreation.
- Work both independently and in groups.
- Work hard, persevere, and act with integrity.

Demonstrate Personal, Social, and Civic Responsibility

- Accept responsibility for one's own behavior and actions.
- Know career options and the academic and occupational requirements needed for employment and economic independence.
- Treat others with respect and understand similarities and differences among people.
- Learn to resolve disagreements, reduce conflict and prevent violence.
- Participate in meaningful community and/or school activities.
- Understand the individual's rights, responsibilities, and roles in the community, state, and nation.
- Understand how the principles of democracy, equality, freedom, law and justice evolve and work in society.
- Analyze, develop, and act on informed opinions about current economic, environmental, political and social issues affecting Massachusetts, United States, and the world.

Glossary

Algorithm A set of rules for solving a problem in a finite number of steps, i.e. long division.

Analytic geometry A branch of mathematics in which algebraic procedures are applied to geometry, i.e. coordinate geometry.

Box-and-whisker plot A method for displaying the median, quartiles, and extremes of a set of data, using a number line.

Calculus The mathematics of change and motion. The main concepts of calculus are limits, derivatives, and areas under curves.

Deductive reasoning A system of reasoning based on definitions and premises.

Discrete mathematics The study of mathematical properties of sets and systems that have a countable number of elements.

Fibonacci numbers A sequence of numbers that begins with 1, and each number that follows is the sum of the previous two numbers, i.e. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.

Function A relation that associates each object in a set with exactly one value.

Inductive reasoning A system of reasoning based on observation and measurement.

Linear equation An equation whose graph is a straight line.

Linear programming A branch of mathematics that uses linear inequalities to solve decision-making problems involving maximums and minimums.

Mathematical structure The framework of mathematics that provides the foundation on which are built the content areas of mathematics.

Matrix, pl. *matrices* A rectangular array of numbers.

Measure of central tendency The mean, median, and mode of a distribution.

Network Collection of points (vertices), some of which may be connected by edges.

Pascal's triangle A triangular arrangement of numbers in which each number in any row is the sum of the two numbers immediately above it.

Proof A method of constructing a valid argument, using deductive reasoning.

Pythagorean Theorem For any right triangle, the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse.

Random variable In statistics, a quantity that takes any of a set of values with specified probabilities.

Rational numbers Set of numbers that can be expressed as the quotient of two integers, in which the denominator is not zero.

Real numbers Set of numbers comprised of rational and irrational numbers.

Sample space The set of all possible outcomes of an experiment.

Scatter plot Two sets of data plotted as ordered pairs in the coordinate plane.

Sieve of Eratosthenes A method of obtaining prime numbers by sifting out multiples of numbers from the set of natural numbers (excluding 1), so the only the prime numbers remain.

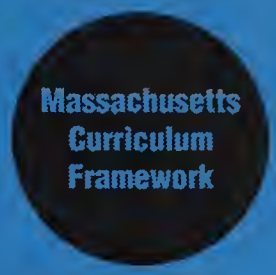
Steiner points Junction points located in such a way that they form the shortest network for a given number of points.

Notes

Notes

Massachusetts Department of Education

This document was prepared by the Massachusetts Department of Education.
Dr. Robert V. Antonucci, Commissioner of Education



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About the Cover: The seven-threaded braid was designed to symbolize the interdisciplinary nature of learning. Each thread represents one of the seven disciplines that comprise the Massachusetts Curriculum Frameworks. These are: The Arts (gold), English Language Arts (green), Health (red), Mathematics (dark blue), Science and Technology (turquoise), Social Studies (orange), and World Languages (purple).